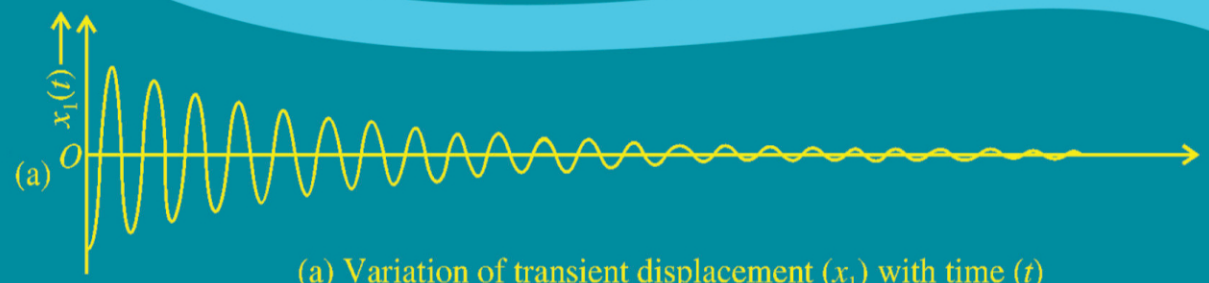
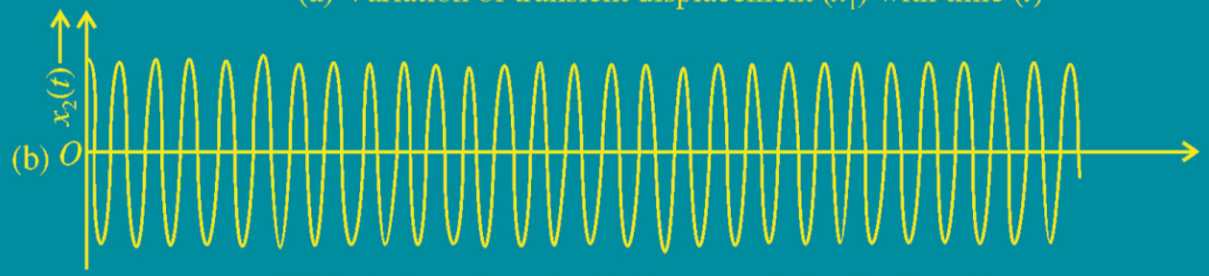
WAVES AND OSCILLATIONS

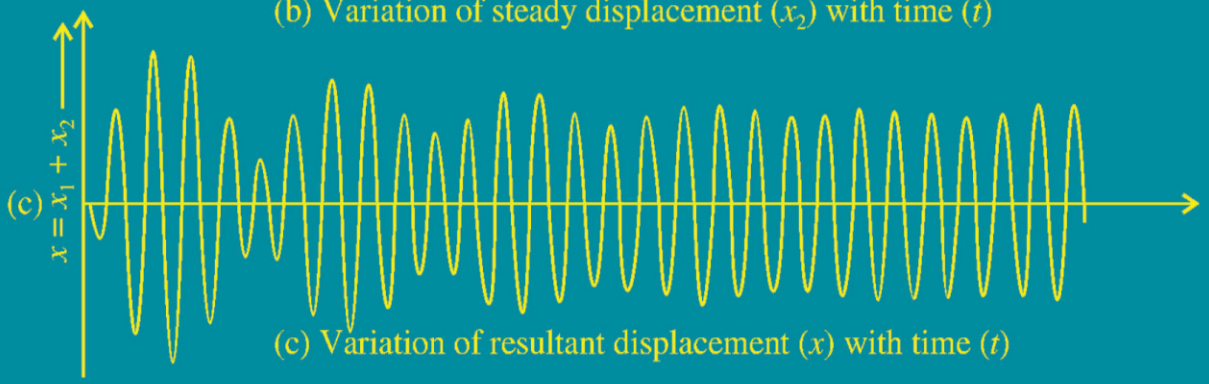
J.C. Upadhyaya



(a) Variation of transient displacement (x_1) with time (t)



(b) Variation of steady displacement (x_2) with time (t)



(c) Variation of resultant displacement (x) with time (t)

Himalaya Publishing House
ISO 9001:2015 CERTIFIED

WAVES AND OSCILLATIONS

Dr. J.C. Upadhyaya
M.Sc., Ph.D., F. Inst. P. (London)
Formerly Director/Professor Incharge, Dau Dayal Institute of Vocational Education,
Dr. B.R. Ambedkar University, AGRA.
Senior Reader in Physics, Agra Collage, AGRA.

Himalaya Publishing House
ISO 9001:2015 CERTIFIED

© **Author**

No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording and/or otherwise without the prior written permission of the publisher.

First Edition : 2016

Published by : Mrs. Meena Pandey for **Himalaya Publishing House Pvt. Ltd.,**
"Ramdoot", Dr. Bhalerao Marg, Girgaon, **Mumbai - 400 004.**
Phone: 022-23860170/23863863, Fax: 022-23877178
E-mail: himpub@bharatmail.co.in; Website: www.himpub.com

Branch Offices :

New Delhi : "Pooja Apartments", 4-B, Murari Lal Street, Ansari Road, Darya Ganj, New Delhi - 110 002. Phone: 011-23270392, 23278631; Fax: 011-23256286

Nagpur : Kundanlal Chandak Industrial Estate, Ghat Road, Nagpur - 440 018. Phone: 0712-2738731, 3296733; Telefax: 0712-2721216

Bengaluru : No. 16/1 (Old 12/1), 1st Floor, Next to Hotel Highlands, Madhava Nagar, Race Course Road, Bengaluru - 560 001. Phone: 080-22286611, 22385461, 4113 8821, 22281541

Hyderabad : No. 3-4-184, Lingampally, Besides Raghavendra Swamy Matham, Kachiguda, Hyderabad - 500 027. Phone: 040-27560041, 27550139

Chennai : New-20, Old-59, Thirumalai Pillai Road, T. Nagar, Chennai - 600 017. Mobile: 9380460419

Pune : First Floor, "Laksha" Apartment, No. 527, Mehunpura, Shaniwarpeth (Near Prabhat Theatre), Pune - 411 030. Phone: 020-24496323/24496333; Mobile: 09370579333

Lucknow : House No 731, Shekhupura Colony, Near B.D. Convent School, Aliganj, Lucknow - 226 022. Phone: 0522-4012353; Mobile: 09307501549

Ahmedabad : 114, "SHAIL", 1st Floor, Opp. Madhu Sudan House, C.G. Road, Navrang Pura, Ahmedabad - 380 009. Phone: 079-26560126; Mobile: 09377088847

Ernakulam : 39/176 (New No: 60/251) 1st Floor, Karikkamuri Road, Ernakulam, Kochi – 682011. Phone: 0484-2378012, 2378016; Mobile: 09387122121

Cuttack : New LIC Colony, Behind Kamala Mandap, Badambadi, Cuttack - 753 012, Odisha. Mob.: 9338746007

Kolkata : 108/4, Beliaghata Main Road, Near ID Hospital, Opp. SBI Bank, Kolkata - 700 010, Phone: 033-32449649, Mobile: 7439040301

DTP by : Chhaya Graphics

Printed at : M/S Sri Sai Art Printer Hyderabad. On behalf of HPH.

PREFACE

The oscillatory and wave phenomena are very common in nature and the relevent study forms a course of extreme importance in basic physics. In the present book on waves and oscillations, the book has been divided into five units with six chapters. First two chapters of the book are devoted to the study of simple, damped and forced harmonic oscillations. Third chapter deals with complex vibrations which are analysed with the help of Fourier theorem. Vibrations of strings and bars are discussed in next two chapters. Last chapter deals with the topic on ultrasonics, being a subject of scientific, teachnologcal and medical importance.

In order to understand the basic concepts of physics, it is necessary that the students have sufficient practice to solve the related problems. To stress this point, the subject comittee has made it necessary that at least three problems are to be asked in each theory paper. This is why we have given a sufficient number of selected informative and modern solved problems inside each chapter followed by a good number of unsloved problems at the end of each chapter.

In writing this book, we have tried to present the subject matter in simple language so that an average student feels interesting in following the text. Though best effort has been made not to have any mistake inside the book, even then some mistakes may have inadvertently crept in. The author shall feel highly grateful to the readers who will point out and inform regarding any mistake. Further any suggestion towards the improvement of the subject matter will be thankfully received.

Date: December 08, 2015 **Dr. J.C. Upadhyaya**

This page is left blank intentionally.

CONTENTS

Chapter	Pages
1. Simple Harmonic Oscillations	**1 – 53**
1. Oscillatoty motion	1
2. Simple harmonic motion and harmonic oscillator	1
3. Differential equation of S.H.M. and its solution	2
4. Energy of harmonic oscillator	5
5. Linear oscillations of spring-mass system	16
6. Angular oscillations	23
(a) Compound pendulum – Determination of g	23
(b) Torsional pendulum	26
7. Additional examples of harmonic oscillator	31
(a) Simple pendulum	31
(b) Charge oscillations in L-C circuit	32
(c) Oscillations of two particles connected by a spring-diatomic molecule	33
8. Principle of superposition	36
9. Superposition of two S.H.M.'s of same frequency along a line	37
10. Combination of two mutually perpendicular harmonic vibrations of same frequency	38
11. Composition of two rectangular S.H.M.'s of nearly equal frequencies	40
12. Composition of two rectangular S.H.M.'s of frequencies in the ratio 2 : 1	40
13. Lissajou's figures	41
14. Superposition of two S.H.M.'s of different frequencies — modulation and beats	44
Questions and Problems	
2. Damped and Forced Oscillations	**54 – 91**
1. Introduction	54
2. Damped harmonic oscillator	54
3. Energy and power dissipation of damped harmonic oscillator-relaxation time	57
4. Quality factor Q	58
5. LCR circuit	59
6. Forced (or driven) harmonic oscillator	67
7. Resonance (amplitude) and sharpness of resonance – half width of resonance curve	71
8. Quality factor and effect of damping	73
9. Velocity resonance	74
10. Power absorption and resonance width	74
11. Driven LCR circuit	76
12. Mechanical and electrical analogues	79
13. Free, damped and forced vibrations – A comparison	80
14. Transient state and steady state	81
Questions and Problems	
3. Complex Vibrations	**92 – 108**
1. Periodic funition	92
2. Fourier's theorem	92
3. Limitations of Fourier's theorem	94
4. Applications of Fourier's theorem	94
(a) Fourier analysis of square wave	95

(b) Fourier analysis of saw tooth wave 99
(c) Fourier analysis of triangular wave 100
5. Importance of Fourier's theorem 103
Questions and Problems

4. Vibrating Strings 109 – 137
1. Wave motion: Transverse and longitudinal waves 109
2. Wave pulse on a string and equation of a wave 109
3. Transverse wave propagation along a stretched string: Velocity of transverse waves 112
4. General solution of the wave equation 113
5. Modes of vibration of stretched string clamped (fixed) at both ends 114
6. Plucked string 117
7. Laws of transverse vibration of strings 118
8. Energy of vibrating string: Energy density 119
9. Energy transport and rate of energy flow 120
10. Transverse impedance 122
11. Reflection and transmission 123
Questions and Problems

5. Vibrations of Bars 138 – 163
1. Introduction 138
2. Longitudinal vibrations in a solid bar – establishment of wave equation and velocity of longitudinal waves 138
3. Wave equation for longitudinal vibrations and its general solution 139
4. Stationary longitudinal waves in bars 140
5. Longitudinal vibrations in a bar, free at both ends (Free-free bar) 141
6. Longitudinal vibrations in a bar, fixed at both ends (Fixed-fixed bar) 143
7. Longitudinal vibrations in a bar fixed at the mid-point 145
8. Longitudinal vibrations in a bar fixed at one end and free at the other (Fixed-free bar) 147
9. Transverse vibrations of a bar – Differential wave equation 148
10. General solution of the wave equation for transverse vibrations 150
11. Transverse vibrations in a bar fixed at one end and free at one the other (Clamped free bar) 151
12. Transverse birations in a bar free at both ends (Free-free bar) 153
13. Velocity of transverse waves in a bar 154
14. Comparision of transverse vibrations of bar and string 155
15. Comparision of logitudinal and transverse vibrations in a bar 155
16. Tunning fork 156
Questions and Problems

6. Ultrasonics 164 – 176
1. Introduction 164
2. Production of ultrasonic waves 164
3. Detection of ultrasonics 169
4. Determination of wavelength and velocity of ultrasonic waves 169
5. Applications of ultrasonic waves 171
Questions and Problems

1

Simple Harmonic Oscillations

1. Oscillatory Motion

Oscillatory or vibratory phenomenon is very common in nature. In physics, familiar examples of this phenomenon are: (i) oscillations of a simple pendulum, (ii) oscillations of spring-mass system, (iii) vibrations of a tuning fork, (iv) vibrations of a guitar string etc. The terms oscillation and vibration have practically the same meaning and both are used for the same purpose. In general usage, an oscillation is possibly slower, but we shall not consider any such distinction. The vibratory behaviour is responsible for all kinds of wave phenomena. In sound waves, particles of the medium and in electromagnetic waves, electric and magnetic field vectors perform oscillatory motions. In fact, the study of oscillatory phenomena is of extreme importance in physics.

The feature that the oscillatory phenomena have in common is periodicity. If a particle moves such that it retraces its path regularly after equal intervals of time, its motion is said to be **periodic**. This inerval of time, required to complete one round trip (cycle) of motion is called **period**. Now, if a particle in periodic motion moves back and forth over the same path about an equilibrium position, we call the motion **vibratory** or **oscillatory**.

The time taken for the system to undergo one complete oscillation is the period T of the system. Since the system executes $1/T$ vibrations per unit time, this quantity is called the frequency of vibration (ν), given by

$$\nu = \frac{1}{T}$$

The dimensions of frequency are $(\text{time})^{-1}$. We sometimes express frequency in cycles (or vibrations) per second. One cycle per second (cps) is known as one hertz (Hz). The **hertz** is the SI unit of frequency.

2. Simple Harmonic Motion and Harmonic Oscillator

Simple harmonic motion is a particular type of periodic motion and is very common in nature. *A system is said to execute* **simple harmonic motion** *when it vibrates periodically in such a way that at any instant the restoring force acting on it is proportional to its displacement from its position of stable equilibrium in its path and is always directed towards that position.*

A system executing simple harmonic motion is called **harmonic oscillator.**

Let us consider a particle of mass m, executing simple harmonic motion about the equilibrium position O with maximum displacement a on either side **[fig. 1(a)]**. At any instant t, if the displacement of the particle is x, then its acceleration will be d^2x/dt^2 and the force (F) will be md^2x/dt^2. Now, according to the definition of simple harmonic motion,

Restoring force (F) $\propto$ –displacement (x)

or $$m\frac{d^2x}{dt^2} \propto -x$$

or $$m\frac{d^2x}{dt^2} = -kx \quad \text{or } F = -\cdot kx \qquad ...(1)$$

where k is the force constant. Here, the negative sign indicates that the force on the particle is directed opposite to x increasing.

The potential energy U of the particle at the displacement x is given by

$$U = -\int_0^x F\,dx$$

$$= \int_0^x kx\,dx = \frac{1}{2}kx^2 \quad ...(2)$$

The variation of force (F) and potential energy (U) of the particle, executing simple harmonic motion, with displacement (x) are shown in **fig. 1(c)** and **fig. 1(b)** respectively. Since U varies as x^2, the potential energy will be the same at the maximum displacement $x_{max} = +a$ or $-a$. This maximum displacement a is the **amplitude** of motion.

Characteristics of Simple Harmonic Motion

Simple harmonic motion is characterized with the following properties:

(1) It is a periodic motion with a certain periodic time T.

(2) The restoring force on the particle is proportional to the displacement from the equilibrium position in a direction opposite to it **[fig. 1(c)].**

(3) The potential energy of the oscillating particle varies as the square of the displacement **[fig. 1(b)]**.

(4) The truning points are situated at equal distances from the equilibrium position.

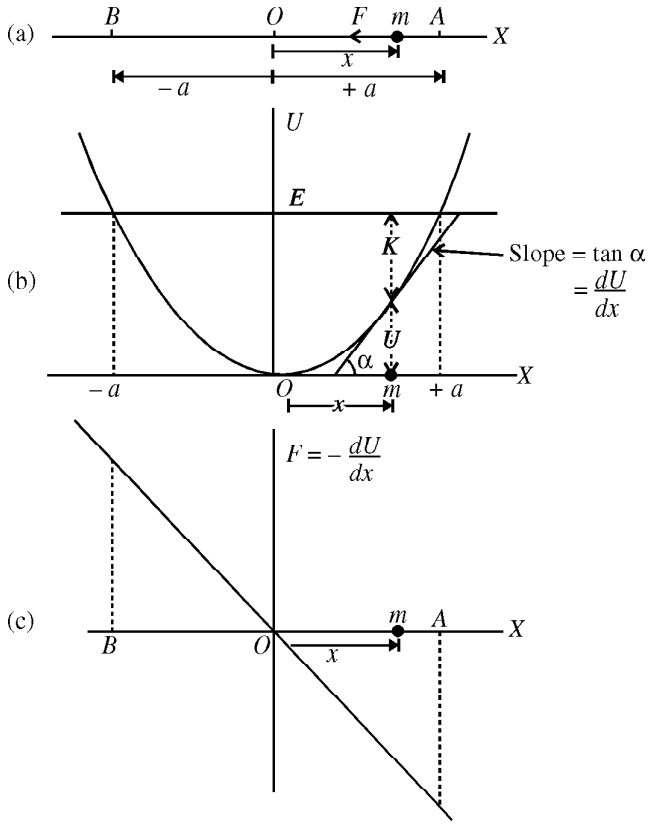


Fig. 1: Simple harmonic motion: (a) Particle displacement between turning points $x = +a$ and $x = -a$, (b) Parabolic potentialenergy curve, (c) Force versus displacement curve (straight line).

3. Differential Equation of Simple Harmonic Motion and its Solution

For simple harmonic oscillations, we have

$$m\frac{d^2x}{dt^2} = -kx \text{ or } \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad ...(1)$$

Let us put $\omega^2 = k/m$. Then

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad ...(2)$$

Eq. (2) is called the **differential equation of simple harmonic motion.**

In order to find the displacement of the particle in S.H.M., we solve eq. (2). Multiplying eq. (2) by $2dx/dt$, we obtain

$$\frac{2dx}{dt}\cdot\frac{d^2x}{dt} + \omega^2 \cdot 2x\frac{dx}{dt} = 0$$

Integrating it, we obtain

$$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 = C, \text{ where } C \text{ is a constant of integration.} \quad ...(3)$$

The velocity of the oscillating particle is zero at the maximum value of the displacement *i.e.*, at $x = x_{max} = a$ (say), $dx/dt = 0$ *i.e.*,

$$0 + \omega^2 a^2 = C \text{ or, } C = \omega^2 a^2.$$

Substituting the value of C in eq. (3), we get

$$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 = \omega^2 a^2 \text{ or } \left(\frac{dx}{dt}\right)^2 = \omega^2(a^2 - x^2)$$

or $$\frac{dx}{dt} = \pm\omega\sqrt{a^2 - x^2} \quad ...(4)$$

If we take positive root, then eq. (4) can be written as

$$\frac{dx}{\sqrt{a^2 - x^2}} = \omega dt \text{ or } \sin^{-1}\left(\frac{x}{a}\right) = \omega t + \phi$$

or $$x = a \sin(\omega t + \phi) \quad ...(5)$$

where ϕ is a constant of integration. If we take negative root, then

$$-\frac{dx}{\sqrt{a^2 - x^2}} = \omega dt \text{ or } \cos^{-1}\left(\frac{x}{a}\right) = \omega t + \varphi$$

or $x = a \cos(\omega t + \varphi)$ where φ is a constant of integration. ...(6)

Eq. (5) and eq. (6), represent sine and cosine forms of the solutions and both are valid solutions of the differential equation (2) of the harmonic oscillator. The two solutions become identical, if the two constants of integration ϕ and φ are related as $\varphi = \phi - \pi/2$. It is to be seen from eqs. (5) and (6) that in simple harmonic motion, the displacement is represented as a sine or cosine function of time and if a graph is plotted between displacement (x) and time (t), it is called a **sinusoidal graph [fig. 2].** In other words, *the motion represented by sinusoidal graph, is known as simple harmonic motion.* In our later analysis, we shall use the sine form of the solution, given by eq. (5).

Simple harmonic motion is characterized with the following terms:

(i) Displacement and amplitude: At any instant t, the **displacement** of a particle, executing simple harmonic motion, is given by

$$x = a \sin(\omega t + \phi) \quad ...(7)$$

When $\sin(\omega t + \phi) = 1$ or -1, the displacement has the maximum value a ($= x_{max}$) on either side of the equilibrium position $x = 0$. Hence a is the **amplitude** of vibration.

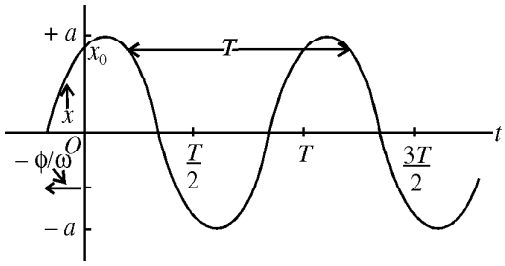


Fig. 2: Simple harmonic motion, $x = a \sin(\omega t + \phi)$, of period T, amplitude a and phase constant ϕ.

(ii) Period and frequency: Now, let us find the **physcial significance** of the constant ω. If the time t in eq. (7) is increased by $2\pi/\omega$, the function becomes

$x = a \sin[\omega(t + 2\pi/\omega) + \phi]$ or $x = a \sin[\omega(t + 2\pi + \phi)$ or $x = a \sin(\omega t + \phi)$.

That is, the displacement of the particle is the same after a time $2\pi/\omega$. Therefore, $2\pi/\omega$ is the **period** (T) of the motion, *i.e.*,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad \left(\because \omega^2 = \frac{k}{m}\right) \quad ...(8)$$

The number of vibrations per second is called the frequency (ν) of the oscillator and is given by

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{m}{k}} \quad ...(9a)$$

$$\therefore \quad \omega = 2\pi\upsilon = \frac{2\pi}{T} \quad ...(9b)$$

The quantity ω is called the **angular frequency**.

(iii) Velocity and acceleration: The displacement of the mass executing S.H.M. is given by $x = a \sin(\omega t + \phi)$

Its velocity (v) is given by

$$v = \frac{dx}{dt} = \omega a \cos(\omega t + \phi) \quad ...(10a)$$

or $$v = \omega a\sqrt{1-\sin^2(\omega t+\phi)}$$

$$= \omega a\sqrt{1-x^2/a^2} = \omega\sqrt{a^2-x^2} \quad ...(10b)$$

Its acceleration is

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 \sin(\omega t + \phi) \quad ...(11a)$$

or $$\frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 x \quad ...(11b)$$

The variations of displacement (x), velocity (dx/dt) and acceleration (d^2x/dt^2) for $\phi = 0$ with time t are given in **fig. 3**. Eqs. (10b) and (11b) give the dependence of velocity and acceleration on the displacement (x). One may observe that the velocity and acceleration lead the displacement in phase by $\pi/2$ and π respectively.

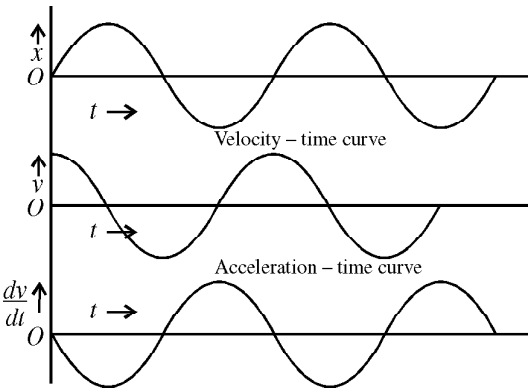


Fig. 3: Variation of displacement (x), velocity (v) and acceleration, with time t.

(iv) Phase: The phase is an important term, used in the theory of vibrations and waves and hence the concept of phase is to be clearly understood.

The term $(\omega t + \phi)$, occurring in the equation (7) for displacement x, is called the **phase** of vibration. The phase tells us the position of the oscillating particle at any time t. At $t = 0$, the value of the phase is ϕ and therefore ϕ is called **initial phase** or **phase constant.** *This gives the information regarding the initial position from where we started to measure the displacement.* Suppose at $t = 0$, the particle is at $x = x_0$. So that from eq. (7), we have

$$x_0 = a \sin\phi \text{ or } \phi = \sin^{-1}(x/a) \quad ...(12)$$

One may note that eq. (2) will be satisfied by eq. (7) for any finite value of a and ϕ. The amplitude a and phase constant ϕ are determined by the initial position and speed of the particle in simple harmonic motion.*

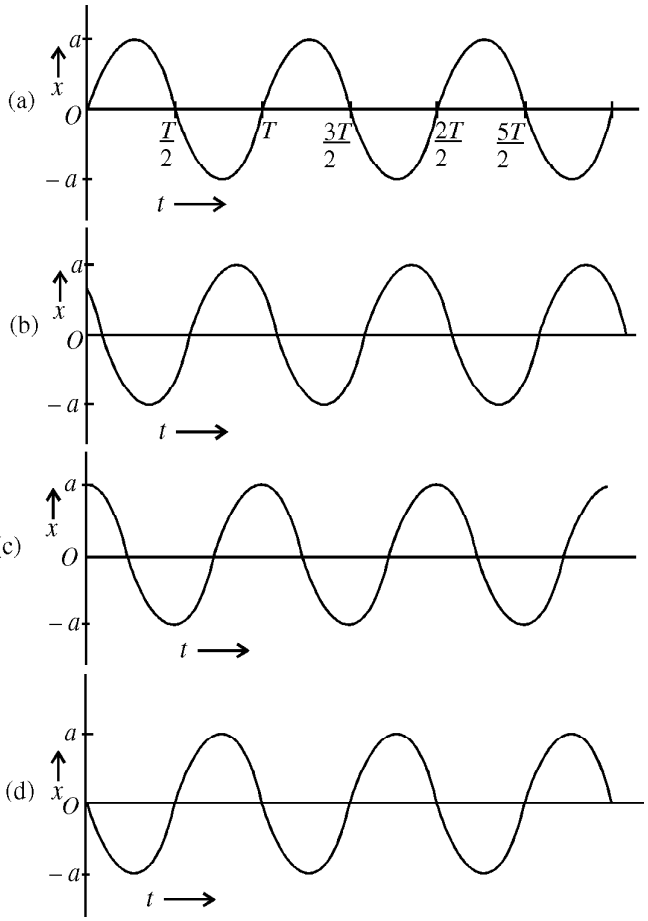


Fig. 4: Displacement-time graphs for S.H.M. represented by $x = a \sin(\omega t + \phi)$: (a) $\phi = 0$, $x_0 = 0$, (b) $\phi = \pi/4$, $x_0 = a/\sqrt{2}$, (c) $\phi = \pi/2$, $x_0 = a$ and (d) $\phi = +\pi$, $x_0 = 0$.

* Initial position (at $t = 0$), $x_0 = a \sin\phi$ and initial speed, $v_0 = \omega a \cos\phi$

$\therefore a^2 \sin^2\phi + a^2\cos^2\phi = x_0^2 + v_0^2/\omega^2$ or $a = \sqrt{x_0^2 + v_0^2/\omega^2}$ and hence $\phi = \sin^{-1}(x_0/a)$.

If we measure the time, when the particle is at the equilibrium position *i.e.*, at $t = 0$, $x = 0$, then from eq. (7), we obtain

$\sin \phi = 0$ or $\phi = 0$

In this situation, eq. (7) is obtained to be

$$x = a \sin \omega t \qquad ...(13)$$

For $\phi = 0$, we have drawn the displacement-time graph in **fig. 4(a)**. For $\phi = \frac{\pi}{4}, \frac{\pi}{2}$ and π the graphs are shown in **fig. 4(b), (c)** and **(d)** respectively. In all cases, the graphs, representing SHM's, have exactly the same shape, but there is the shifting of the origin along the time axis. If **fig. 4(b)** and **fig. 4(c)** represent two separate simple harmonic vibrations, then $\pi/4$ is the phase difference between them and the later is leading in phase. The phase difference for the **fig. 4(a)** and **fig. 4(d)** is π and the two vibrations are said to be in opposite phase.

Alternative Method of Solving the Differential Equation of Harmonic Oscillator

Differential equation of harmonic oscillator is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \text{ where } \omega = \sqrt{\frac{k}{m}}.$$

Let the solution of this equation be $x = Ae^{\alpha t}$

Now, substituting the value of $\frac{d^2x}{dt^2}$ and x in the above equation, we have

$$A\alpha^2 e^{\alpha t} + \omega^2 A e^{\alpha t} = 0, \text{ or } \alpha^2 + \omega^2 = 0, \text{ as in general } Ae^{\alpha t} \neq 0.$$

$$\therefore \quad \alpha = \pm\sqrt{-\omega^2} = \pm i\omega, \text{ where } i = \sqrt{-1}$$

$$\therefore \quad x = A_1 e^{i\omega t} \text{ or } x = A_2 e^{-i\omega t}$$

Hence the general solution of the equation will be given by

$$x = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

where A_1 and A_2 are constants, which can be determined from the initial conditions.

Now, $x = A_1 (\cos \omega t + i \sin \omega t) + A_2 (\cos \omega t - i \sin \omega t)$

$= (A_1 + A_2) \cos \omega t + (A_1 i + A_2 i) \sin \omega t.$

Let $(A_1 + A_2) = a \sin \phi$ and $(A_1 i + A_2 i)\, a \cos \phi$. Then,

$$x = a \sin \phi \cos \omega t + a \cos \phi \sin \omega t$$

or $$x = a \sin (\omega t + \phi)$$

4. Energy of Harmonic Oscillator

A harmonic oscillator possesses two types of energy:

In simple harmonic motion, the oscillating system possesses two types of energy:

(i) **Potential energy,** which is due to its displacement from the mean position.

(ii) **Kinetic energy,** which is due to its velocity.

Hence, at any instant the total energy of the oscillator will be the sum of these two energies. As the system is conservative (*i.e.*, no dissipative or frictional forces are acting), the total mechanical energy E $(= K + U)$ must be conserved.

At the displacement x, the expressions for the potential energy (U), kinetic energy (K) and total energy (E) are given by

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \qquad [\because \omega^2 = k/m] \qquad ...(1a)$$

or $$U = \frac{1}{2}m\omega^2 a^2 \sin^2 (\omega t + \phi) \qquad [\because x = a \sin (\omega t + \phi)] \qquad ...(1b)$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(a^2 - x^2) = \frac{1}{2}k(a^2 - x^2) \quad ...(2a)$$

or $$K = \frac{1}{2}m\omega^2[a^2 - a^2\sin^2(\omega t + \phi)] = \frac{1}{2}m\omega^2 a^2\cos^2(\omega t + \phi) \quad ...(2b)$$

and $$E = U + K = \frac{1}{2}kx^2 + \frac{1}{2}k(a^2 - x^2) = \frac{1}{2}ka^2$$

$$= \frac{1}{2}m\omega^2 a^2 = 2m\pi\nu^2 a^2 \quad (\therefore\ \omega = 2\pi\nu)...(3)$$

Thus, we see that the total mechanical energy is constant, as we expect and has the value $\frac{1}{2}ka^2$, *i.e.*, the total energy is proportional to the square of the **amplitude**. Hence, if the system is once oscillated the motion will continue for indefinite period without any decrease in amplitude, provided no damping (frictional) forces are acting on the system.

It is clear from the expression $U = \frac{1}{2}kx^2$ that if we plot a graph between the potential energy U and displacemetn x, we get a parabola having vertex at x = 0 **[fig. 5]**. The curve for the total energy E (constant) is the horizontal line. The particle cannot go beyond the points where this line intersects the potential energy curve because U can never be larger than E. These points are **turning points** of the motion and correspond to the maximum displacement. At these positions (*i.e.*, at $x = \pm a$) the total energy of the oscillator is wholly potential (i.e., $E = U = \frac{1}{2}ka^2$) but the kinetic energy is zero and so the amplitude of motion is $a = \pm\sqrt{2E/k}$. At the equilibrium position, the potential energy is zero, but the kinetic energy has the mximum value $K_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}ka^2 = E$ with maximum velocity $v_{max} = \sqrt{2E/m}$. At other intermediate points the energy is partly kinetic and partly potential but their sum (total energy) is always $\frac{1}{2}ka^2$.

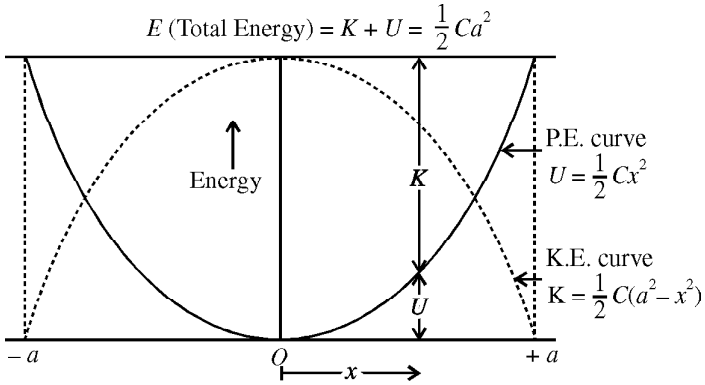


Fig. 5: Variation of potential energy (*U*), kinetic energy (*K*) and total energy (*E*) with displacement (*x*) in simple harmonic motion.

Average values of kinetic and potential energies: The time average of a quantity $Q(t)$ over a time interval T is given by

$$Q_{av} = \frac{1}{T}\int_0^T Q(t)\,dt \quad ...(4)$$

Hence the average kinetic energy for a period T is given by

$$K_{av} = \frac{1}{T}\int_0^T K dt = \frac{1}{T}\int_0^T \frac{1}{2}mv^2\,dt = \frac{1}{T}\int_0^T \frac{1}{2}ma^2\omega^2\cos^2(\omega t + \phi)\,dt \quad \text{[from eq. (2)]}$$

$$= ma^2\frac{\omega^2}{2T}\int_0^{T=2\pi/\omega}\frac{1}{2}[1 + \cos(2\omega t + 2\phi)]\,dx = \frac{1}{4}\frac{ma^2\omega^2}{2\pi/\omega}\cdot\frac{2\pi}{\omega} = \frac{1}{4}ma^2\omega^2 = \frac{1}{4}ka^2 \quad ...(5)$$

Similarly, $$U_{av} = \frac{1}{T}\int_0^T U\,dt = \frac{1}{T}\int_0^T \frac{1}{2}kx^2\,dt = \frac{1}{T}\int_0^T \frac{1}{2}ka^2\sin^2(\omega t + \phi)\,dt = \frac{1}{4}ka^2 \quad ...(6)$$

From eqs. (5) and (6), we get

$$K_{av} = U_{av} \qquad ...(7)$$

Also, $$U_{av} + K_{av} = \frac{1}{2}ka^2 = E = E_{av} \qquad ...(8)$$

because the total energy is constant over the period.

This is an important characteristic of harmonic oscillator that ***its average kinetic energy is equal to the average potential energy***.

We shall consider in the next articles some important examples of simple harmonic motion and the superposition of two simple harmonic motions.

Ex. 1. *A body is executing S.H.M.* and the equation is $x = a \sin(\omega t + \phi)$. If the oscillations are started from the position x_0 with a velocity v_0, show that the amplitude $a = \sqrt{\left[x_0^2 + (v_0/\omega)^2\right]}$ **.(O.U. March 2008)**

Sol. The displacement of a particle executing S.H.M. is given by

$$x = a \sin(\omega t + \phi) \qquad ...(i)$$

Its velocity $$v = \frac{dx}{dt} = \omega a \cos(\omega t + \phi) \qquad ...(ii)$$

Given, at $t = 0$, $x = x_0$ and $v = v_0$. Hence from eq. (i) and (ii), we have

$$x_0 = a \sin \phi \qquad ...(iii)$$

and $$v_0 = \omega a \cos \phi \qquad ...(iv)$$

From (iii), $$\sin \phi = x_0/a \qquad ...(v)$$

From (iv), $$\cos \phi = v_0/\omega a \qquad ...(vi)$$

Squaring (v) and (vi) and then on adding, we obtain

$$\sin^2 \phi + \cos^2 \phi = \frac{x_0^2}{a^2} + \frac{v_0^2}{\omega^2 a^2} \text{ or } 1 = \frac{1}{a^2}\left[x_0^2 + \left(\frac{v_0}{\omega}\right)^2\right]$$

or $$a^2 = \left[x_0^2 + (v_0/\omega)^2\right], \therefore a = \sqrt{x_0^2 + (v_0/\omega)^2}\ .$$

Ex. 2. *A particle of mass* 5 *gm executing simple harmonic motion has amplitude of* 8 *cm. If it makes* 16 *vibrations per sec, find its maximum velocity and energy at mean position.* **(A.N.U. March 2008)**

Sol. Maximum velocity in S.H.M. is given by

$$v_{max} = \omega a \qquad [\because\ v = \omega\sqrt{a^2 - x^2}\ ]$$

Here, $\omega = 2\pi\nu = 2 \times 3{\cdot}14 \times 16$ rad/sec and $a = 8$ cm $= 0{\cdot}08$m

$\therefore$ $v_{max} = 2 \times 3.14 \times 16 \times 0.08 =$ **8.038 m/sec.**

Energy at mean position $= \frac{1}{2}mv_{max}^2 = \frac{1}{2} \times 5 \times 10^{-3} \times (8.038)^2 =$ **0.1615 joule**

Ex. 3. *A particle of mass* 2 *gm moves along the X-axis and is attracted towards origin by a force* 8×10^{-3} *x newton. If is is initially at rest at* $x = 10$ *cm, find* (i) *the differential equation of motion,* (ii) *the position of the particle at any time,* (iii) *the velocity of the particle at any time,* (iv) *amplitude and frequency of vibration.*

Sol. (i) Force $m\frac{d^2x}{dt^2} = -8 \times 10^{-3}x$ or $2 \times 10^{-3}\frac{d^2x}{dt^2} = -8 \times 10^{-3}x$ or $\frac{d^2x}{dt^2} + 4x = 0$

This is the differential equation of motion.

(ii) Let the solution of the equation be $x = a \sin(2t + \phi)$ and hence $\frac{dx}{dt} = 2a \cos(2t + \phi)$

When $t = 0$, $x = 0.1$m, $\therefore\ 0.1 = a \sin \phi$

When $t = 0$, $dx/dt = 0$, $\therefore\ 0 = 2a \cos \phi$, so that $a = 0$ or $\phi = \pi/2$; but $a \neq 0$, $\therefore\ \phi = \pi/2$

Therefore $0.1 = a \sin \pi/2$ or $a = 0.1$ m, $\therefore\ x = 0.1 \sin (2t + \pi/2) = 0.1 \cos 2t$

This equation gives the position of the particle at any time t.

(iii) Now, $v = dx/dt = -0.2 \sin 2t$.

(iv) Amplitude = **0.1 m**; Frequency, $\nu = \omega/2\pi = 2/2\pi = 1/\pi =$ **0.3 Hz.**

Ex. 4. *A particle executing S.H.M. is represented by*

$$y = 10 \sin\left(10t - \frac{\pi}{6}\right),$$

where y is expressed in metres, t in sec. and phase angle in radians. Calculate (i) *the frequency,* (ii) *the time period,* (iii) *the maximum displacement,* (iv) *the maximum velocity,* (v) *the maximum acceleration,* (vi) *displacement, velocity and acceleration at time t* = 0. **(S. K. U. 2011)**

Solution. Equation of S.H.M. is

$$y = a \sin (\omega t + \phi) \qquad ...(i)$$

Comparing eq. (i) with the given equation of S.H.M., we get

$$a = 10 \text{ metre} \quad \text{and} \quad \omega = 10 \text{ sec}^{-1}$$

Now,

(i) Frequency $n = \dfrac{\omega}{2\pi} = \dfrac{10}{2\pi} =$ **1.6 Hz. (nearly)**

(ii) Time period $T = \dfrac{\omega}{2\pi} = \dfrac{2\pi}{10} = 0.63$ sec. (nearly)

(iii) Maximum displacement = $a =$ **10 metre**

(iv) The velocity is given by

$$v = \frac{dy}{dt} = a\omega \cos(\omega t + \phi) \qquad ...(ii)$$

The velocity is maximum when $\cos (\omega t + \phi) = 1$

$$\therefore \quad v_{\max} = \left(\frac{dy}{dt}\right)_{\max} = a\omega = 10 \times 10 = \textbf{100 m/sec}$$

(v) $\therefore$ Acceleration $\left(\dfrac{d^2y}{dt^2}\right) = -a\omega^2 \sin(\omega t + \phi)$ ...(iii)

Acceleration is maximum when $\sin (\omega t + \phi) = 1$ *i.e.*,

$$\left(\frac{d^2y}{dt^2}\right)_{\max} = -a\omega^2 = -10 \times 10\ (10)^2 = \textbf{– 1000 m/sec}^2$$

(vi) At $t = 0$, $y = 10 \sin\left(10 \times 0 - \dfrac{\pi}{6}\right) = 10 \sin\left(-\dfrac{\pi}{6}\right) = -10 \times \left(\dfrac{1}{2}\right) = \textbf{– 5 metre}$

$$v = \frac{dy}{dt} = 10 \times 10 \cos\left(10 \times 0 - \frac{\pi}{6}\right) = 100 \cos\left(-\frac{\pi}{6}\right) = 100 \times 0.866 = \textbf{86.6 ms}^{-1}$$

and $\dfrac{d^2y}{dt^2} = -a\omega^2 \sin\left(10 \times 0 - \dfrac{\pi}{6}\right) = 10 \times 10 \sin\dfrac{\pi}{6} = 1000 \times \dfrac{1}{2} = \textbf{500 ms}^{-2}$

Ex. 5. *The maximum acceleration of a linear harmonic oscillator is x and its maximum speed is y. What is its time period?* **(A.N.U. 2002)**

Sol. In case of S.H.M., $(\text{Acceleration})_{\max} = \omega^2 a = x$ ...(i)

$$(\text{Velocity})_{\max} = \omega a = y \qquad ...(ii)$$

Dividing eq. (i) by eq. (ii), we have

$$\frac{\omega^2 a}{\omega a} = \frac{x}{y} \text{ or } \omega = \frac{x}{y}; \therefore T = \frac{2\pi}{\omega} = \frac{\mathbf{2\pi y}}{\boldsymbol{x}}.$$

Ex. 6. *The displacement equation of a particle describing simple harmonic motion is* $x = 0.01 \sin 50\pi (t + 0.007)$ *metre. Calculate the amplitude, time period, maximum velocity and initial phase for the particle.* **(A.U. 2012; A.N.U. July 2007)**

Sol. Given $x = 0.01\sin 50\pi (t + 0.007)$ ...(i)

Equation of S.H.M. $x = a \sin (\omega t + \phi)$ ...(ii)

Comparing eqs. (i) and (ii),

$$a = \mathbf{0.01} \text{ and } \omega = 50\pi$$

$\therefore$ Time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{50\pi} = \mathbf{0.04 \text{ sec.}}$

Particle velocity $v = \omega\sqrt{a^2 - x^2}$

The velocity is maximum(*i.e.* v_{max}) at $x = 0$ i.e.,

$$v_{max} = \omega a = 50\pi \times 0.01 = 50 \times 3.14 \times 0.01 = \mathbf{1.67 \text{ m/sec.}}$$

Comparision of (i) and (ii) also yields the value of initial phase ϕ i.e.,

$\therefore$ $\phi = 50\pi \times 0.007 = \mathbf{0.35\ \pi \text{ radian}}$

Ex. 7. *The displacement of a particle executing S.H.M. is given by*

$$X = 0.1 \cos \pi\left[t + 1/3\right] \text{ metre.}$$

Calculate the maximum velocity and maximum acceleration of the particle. **(K.U. 2007)**

Sol. Displacement of the particle in S.H.M. is given by

$$x = 0.1 \cos \pi (t + 1/3) \text{ metre}$$

Velocity $v = \frac{dx}{dt} = -0.1 \times \pi \sin \pi (t + 1/3)$

$\therefore$ $v_{max} = 0.1 \times \pi = 0.1 \times 3.14 = \mathbf{0.314 \text{ m/sec.}}$

Acceleration $= \frac{dv}{dt} = -0.1 \times \pi^2 \cos\pi (t + 1/3)$

Maximum acceleration $= 0.1 \times \pi^2 = 0.1 \times (3.14)^2 = \mathbf{0.986 \text{ m/sec}^2.}$

Ex. 8. *A particle under S.H.M. has displacement of* 0.4 *m at the velocity* 0.3 *m/s and a displacement* 0.3 *m at the velocity* 0.4 *m/s. Calculate amplitude and frequency of the oscillation.* **(O.U. 2009)**

Sol. For S.H.M., velocity $v = \omega\sqrt{a^2 - x^2}$

Given, when $x = 0.4$ m, $v = 0.3$ m/sec.

$\therefore$ $0.3 = \omega\sqrt{a^2 - (0.4)^2}$ ...(i)

Also, when $a = 0.3$, $v = 0.4$ m/sec.

$\therefore$ $0.4 = \omega\sqrt{a^2 - (0.3)^2}$ ...(ii)

Squaring (i) and (ii) and dividing

$$\frac{(0.3)^2}{(0.4)^2} = \frac{a^2 - (0.4)^2}{a^2 - (0.3)^2} \text{ or } \frac{9}{16} = \frac{a^2 - 0.16}{a^2 - 0.09}$$

or $9a^2 - 0.81 = 16a^2 - 2.56$ or $16a^2 - 9a^2 = 2.56 - 0.81$

or $7a^2 = 1.75$ or $a^2 = 0.25$ or $a = 0.5$ m

Substituting the value of a in eq. (ii), we get

$$0.4 = \omega\sqrt{(0.25 - 0.09)} = 0.4\omega$$

whence, $\omega = 1$ or $2\pi\, v = 1,\ \therefore\ v = \dfrac{1}{2\pi} = \dfrac{1}{2\times 3.14} =$ **0.16 Hz.**

Ex. 9. *A particle performing* S.H.M. *has a maximum velocity of* 0.4 m/s *and a maximum acceleration of* 0.8 m/sec^2. *Calculate and amplitude and the period of the oscillator.* **(Krishna U. 2014)**

Sol. Given, $v_{max} = \omega a = 0.4$ m/sec.

and $(\text{acc.})_{max} = \omega^2 a = 0.8$ m/sec^2

$\therefore\quad \dfrac{\omega^2 a}{\omega a} = \dfrac{0.8}{0.4}$ or $\omega = 2$ rad/sec.

Period $T = \dfrac{2\pi}{\omega} = \dfrac{2\times 3.14}{2} =$ **3.14 sec.**

Amptitude $a = \dfrac{v_{max}}{\omega} = \dfrac{0.4}{2} =$ **0.2 m.**

Ex. 10. *A particle describing S.H.M. makes* 100 *complete oscillations per minute and its maximum speed is* 5 *m/sec. What is the length of its path and its maximum acceleration? Find the velocity when the particle is half way between its mean position and the extreme position.* **(A.U. April 1995)**

Sol. Frequency $v = 100$ oscillations/min $= \dfrac{100}{60} = \dfrac{5}{3}$ Hz

$\therefore\quad \omega = 2\pi v = 2 \times 3.14 \times \dfrac{5}{3} = 10.47$ rad/sec.

Given, $v_{max} = 5$ m/sec.

But $v_{max} = \omega a,\ \therefore\ 5 = 10.47 \times a$

Hence amplitude $a = \dfrac{5}{10.47} = 0.48$ m

Length of the path of the particle $= 2a = 2 \times 0.48 =$ **0.96 m**

$(\text{acceleration})_{max} = \omega^2 a = (10.47)^2 \times 0.48 =$ **52.62 m/sec^2**

Now, $v = \omega\sqrt{a^2 - x^2}$

For $x = a/2$, $v = \omega\sqrt{a^2 - (a/2)^2} = \omega\sqrt{3a^2/4} = \omega a\sqrt{3}/2$

$= \dfrac{10.47\times 0.48\times 1.732}{2} =$ **4.35 m/sec.**

Ex. 11. *A particle executes S.H.M. with a period of* 0.001 sec *and amplitude* 0.5 cm. *Find its acceleration when it is at* 0.2 *cm away from its mean position and also obtain its maximum velocity.* **(A.N.U. 2014, 02)**

Sol. Acceleration $= -\omega^2 x$

Here, $\omega = \dfrac{2\pi}{T} = \dfrac{2\times 3.14}{0.001}$ and $x = 0.2$ cm $= 0.2 \times 10^{-2}$ m

$\therefore\quad$ Acceleration $= -\left(\dfrac{2\times 3.14}{0.001}\right)^2 \times 0.2 \times 10^{-2} =$ **$-$ 7.89 × 10^4 m/sec^2**

Maximum velocity, $v_{max} = \omega a = \dfrac{2\times 3.14}{0.001} \times 0.5 \times 10^{-2} =$ **31.4 m/sec.**

Ex. 12. *The displacement equation of a particle describing S.H.M. is* $x = 0.5\cos\left(10\pi t + \pi/3\right)$ *m. Calculate (a) amplitude, (b) frequency, (c) phase and (d) displacement after* 1 sec. **(A.N.U. 2001)**

Sol. Standard equation of S.H.M. in cosine form is

$$x = a \cos(\omega t + \phi) \quad \text{...(i)}$$

Given equation of S.H.M.is $x = 0.5 \cos\left(10\pi t + \pi/3\right)$ ...(ii)

Comparing eqs. (i) and (ii), we have

(a) Amplitude a = **0.5 m**

(b) Angular frequency $\omega\theta = 2\pi\nu = 10\pi$, $\therefore$ Frequency $\nu = \dfrac{10\pi}{2\pi}$ = **5Hz.**

(c) Phase (at t = 1 sec.) $= (\omega t + \phi) = 10\pi \times 1 + \dfrac{\pi}{3} = \dfrac{31\pi}{3}$ or $\boldsymbol{\frac{\pi}{3}}$

(d) Displacement x of the particle at time t = 1 sec is [using eq. (i)]

$x = 0.5 \cos\left(10\pi + \dfrac{\pi}{3}\right) = 0.5 \cos\left(\dfrac{31\pi}{3}\right) = 0.5 \cos\dfrac{\pi}{3} = 0.5 \times 0.5 =$ **0.25 metre.**

Ex. 13. *The acceleration of a simple harmonic oscillator is* $4\pi^2$ *m/s*^{2} *at a displacement of* 1 *m. How many oscillations does it make per second?* **(S.V.U. 2002)**

Sol. In case of a harmonic oscillator, accleration = $-\omega^2 x$ or $\omega^2 x$ in magnitude

Here, acceleration = $4\pi^2$ m/sec^2

$\therefore$ $4\pi^2 = \omega^2 \times 1$ or $\omega = 2\pi$

Hence, frequency $\nu = \dfrac{\omega}{2\pi} = \dfrac{2\pi}{2\pi} = 1$ Hz

Thus, the oscillator makes **1 oscillation per second**.

Ex. 14. *The displacement of a linear harmonic oscillator is given by* $x = 4 \sin\left(\dfrac{\pi}{3}t + \dfrac{\pi}{6}\right)$.

Find the period and velocity at t = 1 sec. **(A.U. 2009; A.N.U. 2003)**

Sol. Here, $x = 4 \sin\left(\dfrac{\pi}{3}t + \dfrac{\pi}{6}\right)$ ...(i)

Standard equation of S.H.M. $x = a \sin(\omega t + \phi)$ ...(ii)

Comparing (i) and (ii), we obtain

$$\omega = \frac{\pi}{3} \text{ or } \frac{2\pi}{T} = \frac{\pi}{3} \text{ or } T = \textbf{6 sec.}$$

Also, velocity $v = \dfrac{dx}{dt} = 4 \times \dfrac{\pi}{3} \cos\left(\dfrac{\pi}{3}t + \dfrac{\pi}{6}\right)$

$\therefore$ At t = 1 sec, $v = \dfrac{4\pi}{3}\cos\left(\dfrac{\pi}{3} \times 1 + \dfrac{\pi}{6}\right) = \dfrac{4\pi}{3}\cos\left(\dfrac{3\pi}{6}\right) = \dfrac{4\pi}{3}\cos\left(\dfrac{\pi}{2}\right) = \dfrac{4\pi}{3} \times 0 =$ **0**

Ex. 15. *A body of mass* 0.05 kg *executes simple harmonic motion. When the displacement from the centre of motion is* 0.04 m, *the force acting on the body is* 18×10^{-3} *newton. If the maximum velocity is* 4 *metre/sec., find the amplitude and maximum acceleration.*

Sol. Force, $F = kx = m\omega^2 x$ (in magnitude)

Here, $F = 18 \times 10^{-3}$ N, m = 0.05 kg and x = 0.04 m

$\therefore$ $18 \times 10^{-3} = 0.05 \times \omega^2 \times 0.04$ or $\omega^2 = \dfrac{18 \times 10^{-3}}{0.05 \times 0.04} = 9$ or $\omega = 3$ rad/sec.

Given, $v_{max} = 4$ m/sec. $= \omega a$

$\therefore$ Amptitude $a = \frac{4}{\omega} = \frac{4}{3} =$ **1.33 m.**

$(\text{Acceleration})_{max} = \omega^2 a = 9 \times \frac{4}{3} =$ **12 m/sec^2.**

Ex. 16. *A simple harmonic wave is represented by*

$$y = 10 \sin\left(\frac{2\pi t}{T} + \theta\right)$$

The time period is 30 *sec. When* $t = 0$, *the displacement is* 5 *cm. Find* (*a*) *the phase angle at* $t =$ 7.5 *sec.* (*b*) *the phase difference between two points at a time interval of* 6 *sec.* **(S.V.U. 1995)**

Sol. At $t = 0$, $y = 5$ cm. Hence

$$5 = 10 \sin\theta \text{ or } \sin\theta = \frac{1}{2} \text{ or } \theta = \frac{\pi}{6} \text{ or } 30^\circ$$

Equation of S.H.M. is $y = 10 \sin\left(\frac{2\pi t}{T} + \theta\right)$

(a) Phase angle $\left(\frac{2\pi t}{T} + \theta\right) = \left(\frac{2\pi \times 7.5}{30} + \frac{\pi}{6}\right) = \left(\frac{\pi}{2} + \frac{\pi}{6}\right) = \frac{2\pi}{3}$ or **120°**

(b) If θ_1 and θ_2 be the phase angles corresponding to times t_1 and t_2 respectively, then

$$\theta_1 = \frac{2\pi t_1}{T} + \theta \text{ and } \theta_2 = \frac{2\pi t_2}{T} + \theta$$

$\therefore$ Phase difference $= \theta_2 - \theta_1 = \frac{2\pi}{T}(t_2 - t_1) = \frac{2\pi}{30} \times 6 = \frac{2\pi}{5}$ rad or **72°.**

Ex. 17. *A particle is moving with simple harmonic motion in a straight line, when the distance of the particle from the equilibrium position has the value* x_1 *and* x_2 *the corresponding values of velocity are* u_1 *and* u_2. *Show that the period is* $T = 2\pi\left[\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}\right]^{1/2}$.

Sol. Let the equilibrium position of the particle be *O*. If *P, Q* are the two positions of the particle at distances x_1 and x_2, from *O* respectively **[fig. 6]**, then velocity u_1 of the particle at the position *P* is given by

$$u_1 = \omega\sqrt{a^2 - x_1^2} \quad \text{...(i)}$$

The velocity u_2 at position *Q* is

$$u_2 = \omega\sqrt{a^2 - x_2^2} \quad \text{...(ii)}$$

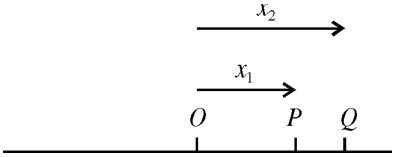


Fig. 6.

Squaring (i) and (ii) and subtracting, we get

$$u_1^2 - u_2^2 = \omega^2(x_2^2 - x_1^2) \text{ or } \omega = \sqrt{\frac{u_1^2 - u_2^2}{x_2^2 - x_1^2}}$$

$\therefore$ time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}}$.

Ex. 18. *A particle moving in simple harmonic motion has speeds* 4 *cm/sec. and* 5 *cm/sec. at distances* 0.5 *cm and* 0.3 *cm respectively from the mean position. Find the amplitude of oscillation.* **(O.U. 2009)**

Sol. Speed $v = \omega\sqrt{a^2 - x^2}$

At $x = 0.5$ cm, $v = 4$ cm/sec., $\therefore\ 4 = \omega\sqrt{a^2 - (0.5)^2}$ ...(i)

At $x = 0.3$ cm, $v = 5$ cm/sec., $\therefore\ 5 = \omega\sqrt{a^2 - (0.3)^2}$ ...(ii)

Squaring (i) and (ii) and dividing

$$\frac{4^2}{5^2} = \frac{a^2 - (0.5)^2}{a^2 - (0.3)^2} \text{ or } \frac{16}{25} = \frac{a^2 - 0.25}{a^2 - 0.09}$$

or $16a^2 - 1.44 = 25a^2 - 6.25$ or $25a^2 - 16a^2 = 6.25 - 1.44$

or $9a^2 = 4.81$ or $a^2 = \frac{4.81}{9}$ or $a =$ **0.73 cm.**

Ex. 19. *If the earth were a homogeneous sphere of radius R, and a straight hole were bored in it through its centre. Show that a particle dropped into the hole will execute a S.H.M. and find out its time period.*

Sol. If the mass of the earth is M and R its radius, then the acceleration due to gravity at the surface of the earth given by

$g = GM/R^2$, where G is gravitational constant.

If the density of the earth is ρ, then $M = \frac{4}{3}\pi R^3 \cdot \rho$.

$\therefore\ g = \frac{4}{3}\pi R\rho \cdot G.$ ...(i)

If g' be the value at depth h below the surface of the earth, then

$$g' = \frac{4}{3}\pi(R-h)\rho \cdot G. \quad \text{...(ii)}$$

Dividing eq. (ii) by eq. (i), we have $\frac{g'}{g} = \frac{R-h}{R}$ or $g' = \frac{g}{R}(R-h)$.

If we measure the distance from the centre of the earth, putting $R - h = x$, acceleration of the particle is,

$$\frac{d^2x}{dt^2} = g' = -\frac{g}{R}x$$

Negative sign shows that the acceleration acts towards the centre of the earth, *i.e.*, opposite to x increasing.

$\therefore\ \frac{d^2x}{dt^2} + \frac{g}{R}x = 0.$

Thus, the particle, dropped into the hole, will execute S.H.M. Its periodic time is given by

$$T = 2\pi\sqrt{\frac{R}{g}}.$$

Ex. 20. *Two cities on the surface of the earth are joined by a straight smooth underground tunnel of length* 640 *km. A body is released into the tunnel from one city. How much time will it take to reach the other city*? *Derive the formula used.* $G = 6.67 \times 10^{-11}$ *S.I. units and* $\rho = 5{,}520$ *kg/m*^{3}. *Calculate the velocity when the body would be nearest to the centre of the earth in its journey.* **(Bombay U.)**

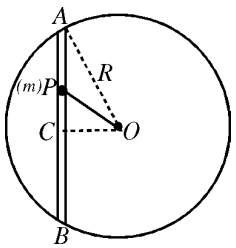


Fig. 7.

Sol. Suppose that A and B are the two cities, which are connected by a straight tunnel of length $AB = 640$ km.

Let O be the centre of the earth and R its radius. OC is the perpendicular, drawn from the centre O on AB, where C is the mid-point of the tunnel AB. Let a body of mass m be released into the tunnel from its end A and let it be at P, distant x from C at any instant t **[fig. 7]**.

If M is the mass of the earth, then on its surface the weight (mg) of the body is given by $mg = GMm/R^2$.

The weight of the body at P, distant OP ($= r$) from the centre of the earth is

$$mg' = \frac{GM'm}{r^2} = \frac{G\frac{4}{3}\pi r^3\rho m}{r^2} = \frac{G\cdot\frac{4}{3}\pi R^3\rho mr}{R^3} = \frac{GMm}{R^2}\cdot\frac{r}{R} \qquad [\because\ M' = \tfrac{4}{3}\pi r^3\rho \text{ and } M = \tfrac{4}{3}\pi R^3\rho]$$

The direction of this force is PO, whose component along PC

$$= \frac{GMm}{R^2}\cdot\frac{r}{R}\cos OPC = \frac{GMm}{R^2}\cdot\frac{r}{R}\cdot\frac{x}{r} = \frac{GMm}{R^3}x$$

Hence the equation of motion of mass m is

$$m\frac{d^2x}{dt^2} + \frac{GMm}{R^3}x = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{GM}{R^3}x = 0.$$

The equation represents a S.H.M. of period $T = 2\pi\sqrt{\frac{R^3}{GM}}$ *i.e.*, this is the time in which the body will reach from A to B and again from B to A. Hence, time taken by the body to pass through the tunnel is

$$\frac{T}{2} = \pi\sqrt{\frac{R^3}{GM}} = \pi\sqrt{\frac{R^3}{G\cdot\frac{4}{3}\pi R^3\rho}} = \sqrt{\frac{3\pi}{4\rho\cdot G}} = \sqrt{\frac{3\times 3.14}{4\times 5520\times 6.62\times 10^{-11}}} = \mathbf{2528\ sec.}$$

Velocity at C, $(v) = \omega a = \sqrt{\frac{GM}{R^3}}\cdot\frac{AB}{2} = \sqrt{G\cdot\frac{4}{3}\pi\rho}\cdot\frac{AB}{2}$

$$= \sqrt{6.67\times 10^{-11}\times\tfrac{4}{3}\times 3.14\times 5520}\cdot\frac{640}{2}\times 10^3 = \mathbf{3.972\times 10^2\ m/sec.}$$

[Energy of Harmonic Oscillator]

Ex. 21. *A particle executing S.H.M. has a maximum displacement 'A' and maximum kinetic energy 'E'. Finds its displacement at the instant its K.E. is* 3E/4. **(O.U. Oct. 2007; A.N.U. 2013)**

Sol. Maximum K.E. $= \frac{1}{2}m\omega^2A^2 = E$

Instantaneous K.E. $= \frac{1}{2}m\omega^2(A^2 - x^2)$

According to the question

$$\frac{1}{2}m\omega^2(A^2 - x^2) = \frac{3E}{4} \quad \text{or} \quad \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{3}{4}\cdot\frac{1}{2}m\omega^2A^2$$

or $$A^2 - x^2 = \frac{3}{4}A^2 \text{ or } x^2 = A^2 - \frac{3A^2}{4} = \frac{A^2}{4}$$

$\therefore$ $$x = \mathbf{A/2.}$$

Ex. 22. *The potential and kinetic energies are same for a S.H. oscillator at a displacement of* 2 cm. *Find its amplitude.* **(S.V.U. 2002)**

Sol. Potential energy $U = \frac{1}{2}kx^2$

Kinetic energy $K = \frac{1}{2}k(a^2 - x^2)$

According to the problem,

$$U = K \text{ i.e., } \frac{1}{2}kx^2 = \frac{1}{2}k(a^2 - x^2) \text{ or } x^2 = a^2 - x^2 \text{ or } a^2 = 2x^2 \text{ or } a = \sqrt{2}\,x$$

Here, x = 2 cm = 0.02 m, $\therefore$ $a = \sqrt{2} \times 0.02 = 1.414 \times 0.02$ m = **2.83 cm.**

Ex. 23. *The total energy of a particle executing S.H.M. is* 0.256 J. *Time period is* 2 *seconds. The displacement of the particle at* (T/4) *sec. is* $8\sqrt{2}$ *cm from the mean position. Calculate the amplitude and mass of the particle.* **(A.N.U. 1994)**

Sol. Equation of S.H.M., $x = a \sin \omega t$

Here, $x = 8\sqrt{2}$ cm = $0.08\sqrt{2}$ m and $\omega t = \frac{2\pi}{T} \times \frac{T}{4} = \frac{\pi}{2}$

$\therefore$ $0.08\sqrt{2} = a \sin \pi/2$ or $a = 0.08\sqrt{2}$ m = $\mathbf{8\sqrt{2}}$ **cm.**

Total energy in S.H.M. is given by

$$E = \frac{1}{2}m\omega^2 a^2$$

Here, E = 0.256 J, $\omega = \frac{2\pi}{T} = \frac{2\pi}{2}$ = 3.14 rad/sec. and $a = 0.08\sqrt{2}$ = 0.113 m

$$\therefore \quad 0.256 = \frac{1}{2} \times m \times (3.14)^2 \times (3.14)^2 \times (0.113)^2$$

whence $$m = \frac{2 \times 0.256}{(3.14)^2 \times (0.113)^2} = \mathbf{4.067\ kg.}$$

Ex. 24. *A particle of mass* 5×10^{-3} kg *gm executes S.H.M. has amplitude of* 0.08 *m. If it makes* 16 *vibrations per* sec. *Find its maximum velocity and energy at mean position.*

(S.U. 2014; A.N.U. 2009; V.S.U. 2014)

Sol. Maximum velocity $v_{max} = \omega a = 2\pi v a$

Here, v = 16 Hz and a = 8 cm = 0.08 m

$$v_{max} = 2 \times 3.14 \times 10 \times 0.08 = \mathbf{8.038\ m/sec.}$$

The energy at mean position is entirely in kinetic form. Hence

$$E = K_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2} \times 5 \times 10^{-5} \times (8.038)^2 = \mathbf{0.16\ joule.}$$

Ex. 25. *The total energy of a particle executing a S.H.M. of period* 2π *second is* 10.24×10^{-4} *joule. The displacement of the particle at* $\pi/4$ *sec. is* $8\sqrt{2}$ *cm. Calculate the amplitude of the motion and the mass of the particle.*

Sol. The displacement of a particle, executing S.H.M. is given by

$$x = a \sin \pi t = a \sin 2\pi t/T \qquad [\because\ \omega = 2\pi/T]$$

$$\therefore \quad 8\sqrt{2} \times 10^{-2} = a \sin \frac{2\pi}{2\pi} \times \frac{\pi}{4} = a \sin \frac{\pi}{4} = \frac{a}{\sqrt{2}}$$

$\therefore$ Amplitude of vibration a = 0.16 m.

If m be the mass of the particle, then its total energy is given by

$$E = \frac{2m\pi^2 a^2}{T^2} = 10.24 \times 10^{-4} \text{ or } m = \frac{10.24 \times 10^{-4} \times T^2}{2\pi^2 \times a^2}$$

Here $T = 2\pi$ sec and a = 0.16 m, $\therefore$ $m = \frac{10.24 \times 10^{-4} \times 2\pi \times 2\pi}{2 \times \pi^2 \times 0.16 \times 0.16} = \mathbf{0.8\ kg.}$

Ex. 26. *If the potential energy of a harmonic oscillator in its resting position is* 5 *joules and the total energy is* 9 *joules, when the amplitude is* 1 *metre, what is the force constant*? *If its mass is* 2 *kg, what is the period*?

Sol. Potential energy in rest position = 5 joules.

At the maximum displacement (equal to amplitude), the potential energy = 9 joules.

$\therefore$ Gain in potential energy = 9 – 5 = 4 joules.

The gain in energy at the maximum displacement should be equal to $\frac{1}{2}kx_{\max}^2 = \frac{1}{2}k \times (1)^2 = k/2$ joules. Thus,

$$k/2 = 4 \quad \text{or} \quad k = \mathbf{8\ joules/m.}$$

$\therefore$ Period $T = 2\pi\sqrt{m/k} = 2\pi\sqrt{2/8} = \mathbf{3.14\ sec.}$

Ex. 27. *What fraction of total energy is kinetic and what fraction is potential when the displacement is one-third of amplitude?* **(S.V.U. 2010; Y.V.U. 2014)**

Sol. Given, displacement $x = a/2$.

Total energy $E = \frac{1}{2}ka^2$, K.E. $= \frac{1}{2}k(a^2 - x^2)$ and P.E. $= \frac{1}{2}kx^2$

$$\therefore \quad \frac{\text{Kinetic energy}}{\text{Total energy}} = \frac{\frac{1}{2}k(a^2 - x^2)}{\frac{1}{2}ka^2} = \frac{a^2 - (a/2)^2}{a^2} = \frac{a^2 - a^2/4}{a^2} = \mathbf{\frac{3}{4}}$$

and $$\frac{\text{Potential energy}}{\text{Total energy}} = \frac{\frac{1}{2}kx^2}{\frac{1}{2}ka^2} = \frac{x^2}{a^2} = \frac{(a/2)^2}{a^2} = \mathbf{\frac{1}{4}}.$$

Ex. 28. *Prove that in simple harmonic motion when the average is taken with respect to the position over one cycle, the average potential energy equals* $\frac{1}{6}ka^2$ *and the average kinetic energy equals* $\frac{1}{3}ka^2$. *Explain physically why this result is different to that when the average is taken with respect to time.*

Sol. $U = \frac{1}{2}kx^2$ and $K = \frac{1}{2}k(a^2 - x^2)$

$\therefore$ $$U_{av} = \frac{1}{a}\int_0^a \frac{1}{2}kx^2 dx = \frac{k}{2a}\left(\frac{x^3}{3}\right)_0^a = \frac{1}{6}ka^2$$

and $$K_{av} = \frac{1}{a}\int_0^a \frac{1}{2}k(a^2 - x^2)dx = \frac{k}{2a}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{1}{3}ka^2$$

5. Linear Oscillations of Spring-mass System

The system consists of a massless spring, one of its ends is connected to a mass m and the other end is fixed to a rigid support. Here, the mass and spring are on a smooth horizontal surface **[fig. 8]**. If a force is applied on the mass to stretch or compress the spring and then released, the motion of the mass is simple harmonic. If x is the displacement of mass m at any time, then the restoring force exerted by the spring is

$F_x = -kx$, where k is force constant.

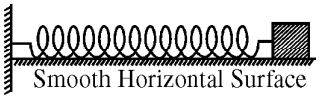


Fig. 8.

Also, force $= m(d^2x/dt^2)$

Therefore, equation of motion is

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \text{ or } \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad ...(1)$$

where $\omega = \sqrt{k/m}$.

This equation represents that the motion of spring-mass system is simple harmonic. Its periodic time and frequency is given by

$$T = 2\pi\sqrt{\frac{m}{k}}\ ;\ n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad ...(2)$$

The solution of the equation of motion is

$$x = a \sin(\omega t + \phi), \text{ where } \omega = \sqrt{\frac{k}{m}} \qquad ...(3)$$

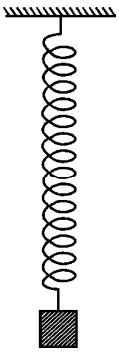


Fig. 9.

Now, suppose the spring-mass system is hanging vertically from a rigid support and it is oscillating in earth's gravitational field. **[Fig. 9]**.

The question arises: Is there any effect of gravity on the oscillations of the hanging spring-mass arrangement? For this purpose, first we consider a spring of negligible mass, suspended vertically from a rigid support *S* **[fig. 10(a)].** To the free end, now we attach an object of mass *m*. Due to the weight *mg* of the object, the spring extends and suppose that the equilibrium position is attained at an extension x_0 of the spring **[fig. 10 (b)]**. At equilibrium, the spring force ($-kx_0$) balances the weight (*mg*) of the object and the net force is zero *i.e.*,

$$mg - kx_0 = 0 \qquad ...(4)$$

Now if mass is displaced downwards through distance *x* from this equilibrium position *O*, the total spring force will be $-k(x_0 + x)$. Hence the net force is $mg - k(x_0 + x) = -kx$, which tends to restore the object to the equilibrium position *O* **[fig. 10(c)]**. Thus the restoring force is $F = -kx$ and the equation of motion is

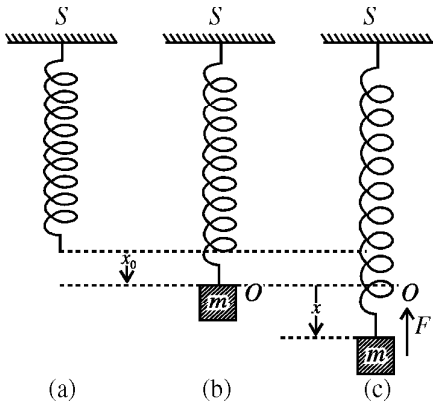


Fig. 10: (a) Spring hanging from a rigid support S; (b) Equilibrium position of the mass *m*, attached to the free end of the spring: $mg - Cx_0 = 0$; (c) Oscillating mass under the restoring force $F = -Cx$.

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad ...(5)$$

This is the standard differential equation of simple harmonic motion whose frequency of vibration is given by eq. (2). Thus the gravity has no effect on the force constant or frequency of oscillation. It is to be noted that the oscillations are vertical due to the vertical tension in the spring, caused by the weight of the object.

Ex. 1. *A spring is stretched by* 8 cm *by a force of* 10 *newton. Find the force constant of the spring. What will be the period of a* 4 *kg mass suspended by it?* **(A.N.U. 2010, 02)**

Sol. If a force *F* stretched a spring by *x*, then in equilibrium

$F = kx$, where *k* is force constant of the spring.

Here, $F = 10$ N and $x = 8$ cm $= 0.08$ m.

$\therefore$ Force constant $k = \dfrac{F}{x} = \dfrac{F}{0.08} =$ **125 N/m**

Period $T = 2\pi\sqrt{\dfrac{m}{k}} = 2 \times 3.14\ \sqrt{\dfrac{4}{125}} =$ **1.123 sec.**

Ex. 2. *A body of* 0.5 kg *mass is hanged to a spring and made to oscillate. For time* $t = 0$. *displacement is* 0.44 m. *acceleration is* – 0.176 m/sec^2. *Find the force constant of spring.* **(K. U. 2013)**

Sol. Acceleration, $a = \omega^2 x = \frac{k}{m} \times x \qquad \left(\because \omega^2 = \frac{k}{m}\right)$

Hence $k = \frac{ma}{x} = \frac{0.5 \times 0.0176}{0.44} =$ **0.2 N/m**

Ex. 3. *A* 1.6 *kg-wt. extends a spring* 8 *cm from its unstretched position. The mass is replaced by a body of* 50 *gm. The mass is pulled and then released, find the period of oscillation.*

Sol. $F = kx$, or $k = F/x$

$\therefore \quad k = \frac{1.6g}{0.08} = \frac{1.6 \times 9.8}{0.08} =$ **196 N/m**

$\therefore \quad$ Period, $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{50 \times 10^{-3}}{196}} =$ **0.11 sec.**

Ex. 4. *Aspring is stretched by a mass of* 1 kg. *If the extension is* 10 cm, *calculate the force constant.* **(A.U. 2012)**

Sol. According to the problem, the force $F = mg = 1 \times 9.8$ N extends the spring by $x_0 = 0.1$ m. This means that

$$mg = kx_0 \textit{ i.e.,} \; 1 \times 9.8 = k \times 0.1$$

Hence force constant $k = \frac{9.8}{0.1} =$ **9.8 N/m**

Ex. 5. *A spring has force constant k and mass m hanging from its end. It is oscillating with a frequency f. If mass is doubled, how the frequency changes?* **(K.U. 2014)**

Sol. Frequency $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ ...(i)

If m is replaced by $2m$, the frequency (f') is

$$f' = \frac{1}{2\pi}\sqrt{\frac{k}{2m}} \qquad \text{...(ii)}$$

Diving (ii) by (i), we get

$$\frac{f'}{f} = \sqrt{\frac{k}{2m}} \times \sqrt{\frac{m}{k}} = \frac{1}{\sqrt{2}} \text{ or } f' = \frac{f}{\sqrt{2}} \qquad \text{...(iii)}$$

Ex. 6. *A body of mass* 0.2 kg *is suspended from a spring, the spring extends by* 0.1 *m. The velocity of the body is* 0.4 *m/sec, when the displacement is* 0.05 *m. Find the time period, frequency and amplitude* **(A.N.U. 2001)**

Sol. $F = mg = kx$ or $k = \frac{mg}{x}$

Here, $m = 0.2$ kg, $g = 9.8$ m/sec.^{2} and $x = 0.1$ m

$\therefore \quad k = \frac{0.2 \times 9.8}{0.1} = 19.7$ N/m

Period, $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.2}{19.6}} =$ **0.634.**

Frequency, $\nu = \frac{1}{T} = \frac{1}{0{\cdot}634} = \mathbf{1{\cdot}577 Hz}$

Also, velocity $v = \omega\sqrt{a^2 - x^2}$

Here, v = 0·4m/sec, $\omega = 2\pi\nu = 2\pi \times 1.577 = 9.897$ rad/sec. and x = 0.05 m.

$\therefore \quad 0.4 = 9.897\sqrt{a^2 - (0{\cdot}05)^2}$ or, $(0.4)^2 = (9.897)^2\,[(a^2 - (0.05)^2]$ or $a^2 = (0.05)^2 + \frac{(0.4)^2}{(9.897)^2}$

whence $a = \mathbf{0{\cdot}0643 m.}$

Ex. 7. *A body of mass* 4.9 *kg hangs from a spring and oscillates with a period of* 0.6 *sec. How much will the spring shorten when the body is removed?* **(A.N.U. July 2007)**

Sol. For spring-mass system period $T = 2\pi\sqrt{\frac{m}{k}}$

Here, m = 4.9 kg and T = 0.6 sec

$\therefore \qquad 0.6 = 2\pi\sqrt{\frac{4.9}{k}}$,

whence $k = \frac{4\pi^2 \times 4.9}{(0.6)^2} = \frac{4 \times (3.14)^2}{(0.6)^2} \times 4.9 = 536.8$ N/m

Due to the weight of the mass let the spring be extended by x_0 i.e., $mg = kx_0$

$\therefore \qquad x_0 = \frac{mg}{k} = \frac{4.9 \times 9.8}{536.8} = \mathbf{0.089\ m = 8.9\ cm.}$

Ex. 8. *A spring is strectched by* 8 cm *by a force of* 10 N. *Find its force constant. What will be the frequency of a* 4 kg *mass suspended from it?* **(A.N.U. 2012)**

Sol. Frequency $\nu_1 = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)}$

where k is force constant.

From the relation, $F = kx$,

$$k = \frac{F}{x} = \frac{10}{0.08} = 125 \text{ N/m}$$

Here, x = 8 cm = 0.08 m and = 4 kg

$\therefore$ Frequency $\nu = \frac{1}{2 \times 3.14} \times \sqrt{\frac{125}{4}} = \mathbf{0.9\ Hz}$

Ex. 9. *The length of a weightless spring increases by* 2 *cm when a weight of* 1.0 *kg is suspended from it. The weight is pulled down by* 10 *cm and released. Determine (i) period of oscillation (ii) total energy of oscillation of spring mass sytem.*

Sol. $k = \frac{F}{x} = \frac{1.0 \times 9.8}{0.02} = 490$ N/m

$\therefore \qquad$ Period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1.0}{490}} = \mathbf{0.29\ sec.}$

$$\text{Total energy} = \frac{1}{2}ka^2 = \frac{1}{2} \times 490 \times (0.1)^2 = \mathbf{2.45\ joule.}$$

Ex. 10. *A mass of* 3 *kg is hung from a vertical spring. When a mass of* 0.5 *kg is gently added the spring is further stretched by* 5 *cm. If the extra mass is removed and the first mass is set into oscillation, calculate the period of oscillation.* **(A.N.U. 1997)**

Sol. Force constant $k = \frac{F}{x} = \frac{mg}{x} = \frac{0.5 \times 9.8}{5 \times 10^{-2}} = 98$ N/m

Period $T = 2\pi\sqrt{\frac{m}{k}} = 2 \times 3.14 \sqrt{\frac{3}{98}} =$ **1.1 sec.**

Ex. 11. *A body of mass* 0.1 *kg hanging from a spring is oscillating with a period of* 0.2 *second and an ampliude of* 1 *metre. Find the maximum value of the force, acting on the body and the value of the force-constant of spring.* **(A.N.U. 2009)**

Sol.: Period $T = 2\pi\sqrt{\left(\frac{m}{k}\right)}$

Here, $T = 0.2$ sec. and $m = 0.1$ kg,

$\therefore \quad 0.2 = 2\pi\sqrt{\left(\frac{0.1}{k}\right)}$ or $0.04 = 4\pi^2\left(\frac{0.1}{k}\right)$ or $k = \frac{0.4\pi^2}{0.04} =$ **$10\pi^2$ N/m**

Now, Maximum force = $m \times$ Maximum acc. = $m\omega^2 a$,

Here, $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi$ and $a = 1m$

$\therefore$ Maximum force = $(0.1) \times (10\pi)^2 \times 1 =$ **9.87 N**

Ex. 12. *The period of oscillation of a spring is* 0.1 *second. The amplitude of oscillation is* 2 *cm. A body of mass* 0.2 *kg is hung from it. Calculate the maximum P.E. that will be stored in it.*

(S.V.V. 2009; A.N.U. 1996)

Sol. Maximum P.E. $= \frac{1}{2}m\omega^2a^2 = \frac{1}{2}m\left(\frac{2\pi}{T}\right)^2 a^2 = \frac{2\pi^2 ma^2}{T^2}$ $\left(\because \omega = \frac{2\pi}{T}\right)$

$= \frac{2 \times (3.14)^2 \times (0.2) \times (0.02)^2}{(0.1)^2} =$ **0.158 joule.**

Ex. 13. Fig. 11 *shows a mass M resting on a smooth table between two firm supports A, B and controlled by two massless springs. If the mass M is* 30 *gm, and the force constants of the two springs are* 2 *and* 1 *N/m, deduce* (i) *the frequency of small oscillations of M,* (ii) *the enrgy of oscillations for amplitude* 0.5 *cm.*

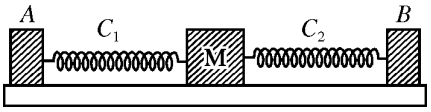


Fig. 11.

Sol. If the mass M is displaced through a distance x towards left or right, the restoring forces in the two springs are

$$F_1 = -k_1x \text{ and } F_2 = -k_2x$$

acting on the mass along the same direction. So that, the total restoring force on the mass M is $F = -(k_1 + k_2)x$. Thus, the effective force constant of the system is $k = k_1 + k_2 = 2 + 1 = 3$ N/m.

(i) Mass $M = 30$ gm $= 0.03$ kg

$\therefore$ Frequency, $\upsilon = \frac{1}{2\pi}\sqrt{\frac{k}{M}} = \frac{1}{2\pi}\sqrt{\frac{3}{0.03}} = \frac{5}{\pi} = \frac{5}{3.14} =$ **1.6 (nearly).**

(ii) Energy E = Max. P.E. $= \frac{1}{2}kx_{max}^2 = \frac{1}{2} \times 3 \times (0{\cdot}5 \times 10^{-2})^2 =$ **3.75×10^{-5} J.**

Ex. 14. (a) *Two springs of force constant k_1 and k_2 are connected in series. What will be the effective force constant of the system.*

(A.N.U. March 2007)

(b) *If M = 30 gm, k_1 = 2 N/m and k_2 = 1 N/m, find the effective force constant and period of oscillation.*

Sol. (a) When the springs of force constant k_1 and k_2 are connected as shown in **fig. 12**, the displacement x of mass M produces the same restoring force F in the two springs. If the two springs are extended through x_1 and x_2, then

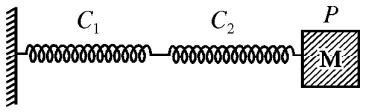


Fig. 12.

$$F = -kx_1 = -kx_2$$

and therefore $$x = x_1 + x_2 = -\frac{F}{k_1} - \frac{F}{k_2} = -F\left(\frac{k_1+k_2}{k_1k_2}\right), \quad \therefore \quad F = -\frac{k_1k_2}{k_1+k_2}x$$

Hence force constant, $k = \dfrac{k_1k_2}{k_1+k_2}$.

(b) Here, k_1 = 2 N/m, k_2 = 1 N/m and M = 30 gm = 0.03 kg

$\therefore$ Effective force constant $k = \dfrac{k_1k_2}{k_1+k_2} = \dfrac{2\times1}{2+1} = \mathbf{\frac{2}{3}\ N/m}$

Period of oscillation $T = 2\pi\sqrt{\dfrac{M}{k}} = 2\pi\sqrt{\dfrac{0.03\times3}{2}} = \mathbf{1.33\ sec.}$

Ex. 15. Spring having mass: *Find the period and frequency of oscillation of a body of mass M suspended from a uniform spring of force constant k and mass m is given by* $n = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{M+m/3}}$.

(K.U. 2005)

Sol. If l is the length of the spring, then mass per unit length = m/l.

Now, let us consider a length ds of the spring at a distance s (measured along the length) below the fixed upper end **[fig. 13]**. The mass of this small element is $(m/l)\ ds$.

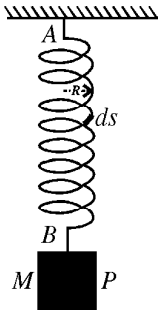


Fig. 13.

If the velocity of the lowest point of the spring is v, then evidently the velocity of the length ds at a distance s from rigid support will be $(s/l)v$.

$\therefore$ Kinetic energy of the element $= \dfrac{1}{2}\left(\dfrac{m}{l}ds\right)\left(\dfrac{vs}{l}\right)^2$

$\therefore$ K.E. of the whole spring $= \displaystyle\int_0^l \frac{1}{2}\frac{mv^2}{l^3}s^2ds = \frac{1}{2}\frac{mv^2}{l^3}\cdot\frac{l^3}{3} = \frac{mv^2}{6}$

Kinetic energy of the mass M at the lower end of the spring $= \dfrac{1}{2}Mv^2$.

Total kinetic energy of the system $= \dfrac{1}{2}Mv^2 + \dfrac{1}{6}mv^2 = \dfrac{v^2}{2}\left(M+\dfrac{m}{3}\right)$

If x is the displacement of the mass from its mean position, then $v = \dfrac{dx}{dt}$

and total K.E. $= \left(M+\dfrac{m}{3}\right)\left(\dfrac{dx}{dt}\right)^2$.

Potential energy of the spring $= \displaystyle\int_0^x kx\,dx = \frac{1}{2}kx^2$.

From the law of conservation of energy

$$\frac{1}{2}\left(M+\frac{m}{3}\right)\left(\frac{dx}{dt}\right)^2+\frac{1}{2}kx^2 \;=\; \text{a constant.}$$

Differentiating it with respect to time and dividing by *dx*/*dt*, we have

$$\left(M+\frac{m}{3}\right)\frac{d^2x}{dt^2}+kx \;=\; 0 \quad \text{or} \quad \frac{d^2x}{dt^2}+\frac{k}{(M+m/3)}x \;=\; 0$$

This is the equation of simple harmonic motion, whose time period and frequency are given by

$$T \;=\; 2\pi\sqrt{\frac{(M+m/3)}{k}} \;\text{ and }\; \upsilon \;=\; \frac{1}{T} \;=\; \frac{1}{2\pi}\sqrt{\frac{k}{(M+m/3)}}\,.$$

Ex. 16. *A body of mass* 0.095 *kg hanging from a spring of mass* 0.015 *kg oscillates up and down simple harmonically. The force constant of the spring is* $10\pi^2$ *N/m. Calculate the frequency of oscillations. The spring is now replaced with a heavier one but the same spring constant. How does the frequency change?* **(O.U. 2002)**

Sol. The frequency of oscillation of a spring, having mass *m* and loaded with a mass *M*, is given by

$$\upsilon = \frac{1}{2\pi}\sqrt{\frac{k}{M+(m/3)}} \;=\; \frac{1}{2\pi}\;\sqrt{\frac{10\pi^2}{0.095+(0.015/3)}} \;=\; \frac{1}{2\pi}\;\sqrt{\frac{10\pi^2}{0.1}} \;=\; \frac{10\pi}{2\pi} \;=\; \textbf{5 Hz.}$$

If the spring is replaced with a heavier one, the mass *m* will increase and hence the frequency υ will decrease.

Ex. 17. *In a vertical spring-mass system, the perod of the oscillation is* 0.26 second *when the mass is* 0.4 kg *and period becomes* 0.34 second *when an additional mass of* 0.3 kg *is added. Calculate the mass of the spring and the force constant of the spring.* **(A.N.U. 1994; S.V.U. 1995)**

Sol. When a mass *M* is suspended from the free end of a spring, having mass *m*, the perid of oscillation is given by

$$T \;=\; 2\pi\sqrt{\frac{M+(m/3)}{k}}$$

where *k* is force constant.

Here, M = 0.4 kg, T = 0.26 second and when M = 0.7 kg, T = 0.34 second.

$$\therefore \qquad 0.26 \;=\; 2\pi\;\sqrt{\frac{0.4+(m/3)}{k}} \;\text{ or, }\; \frac{(0.26)^2}{0.4+(m/3)} \;=\; \frac{4\pi^2}{k} \qquad \text{...(i)}$$

$$\text{Similarly,} \qquad 0.34 \;=\; 2\pi\;\sqrt{\frac{0.7+(m/3)}{k}} \;\text{ or }\; \frac{(0.34)^2}{0.7+(m/3)} \;=\; \frac{4\pi^2}{k} \qquad \text{...(ii)}$$

From eqs. (i) and (ii), we have

$$\frac{(0.26)^2}{0.4+(m/3)} \;=\; \frac{(0.34)^2}{0.7+(m/3)} \;\text{ or }\; \frac{0.7+(m/3)}{0.4+(m/3)} \;=\; \left(\frac{0.34}{0.26}\right)^2 \;=\; 1.71$$

$$\therefore \qquad 0.7 + (m/3) \;=\; 0.684 + (m/3) \times 1.71$$

$$\text{or} \qquad \frac{m}{3}\;(1.71-1) \;=\; 0.7 - 0.684 \text{ or } m \;=\; 3 \times \frac{0.016}{0.71} \;=\; \textbf{0.676 kg.}$$

The force constant can be obtained by putting the value of *m* in (i) i.e.,

$$\frac{(0.26)^2}{0.4+(0.0676/3)} \;=\; \frac{4\pi^2}{k} \;=\; \frac{4(3.14)^2}{k} \;\text{ or }\; \frac{0.0676}{0.4225} \;=\; \frac{39.44}{k}$$

$$\therefore \qquad k \;=\; 39.44 \times \frac{0.4225}{0.0676} \;=\; \textbf{246.5 N/m.}$$

6. Angular Oscillations — Compound and Torsional Pendulums

In case of spring-mass system, the mass oscillates about an equilibrium position along a straight line. When we observe the oscillations of a pendulum or balance wheel in a watch, the system rotates back and forth about an equilibrium position under the action of restoring torque. In these rotational systems, the restoring torque and moment of inertia replace restoring force and mass of spring-mass system respectively. If the restoring torque in the system is proportional to the angular displacement and directed opposite to it, the system executes **angular simple harmonic vibrations** and a differential equation analogous to the equation of harmonic oscillator can be established.We consider here the restoring torques due to gravity (*e.g.*, in compound pendulum) and due to elasticity (*e.g.*, in torsional pendulum).

(a) Compound Pendulum – Determination of *g*

A compound pendulum is a rigid body capable of oscillating freely in a vertical plane about a horizontal axis under the action of gravity. Let **fig. 14** represent the vertical section of a rigid body through the centre of gravity *G*. It is capable of oscillating about a horizontal axis through *S*, the centre of suspension, where the vertical section through *G* meets the horizontal axis. In equilibrium, the centre of gravity *G* of the body lies vertically below *S*. Let the distance *SG* = *l*. Now if the body is displaced to one side and left free, it oscillates about the equilibrium position. If θ be the angular displacement at any instant *t*, then the restoring torque about *S* is given by

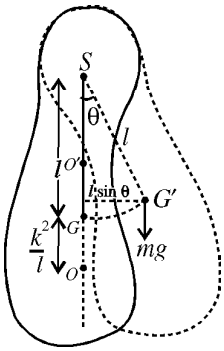


Fig. 14: Compound pendulum.

$$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta \qquad ...(1)$$

where *I* is the moment of inertia of the body about the horizontal axis through *S* and *mg* is the weight of the body.

For sufficiently small angular displacement θ, eq. (1) assumes the form

$$\frac{d^2\theta}{dt^2} + \frac{mgl}{I}\theta = 0$$

or
$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad ...(2)$$

where
$$\omega = \sqrt{mgl/I}\,.$$

Eq. (2) represents that the compound pendulum is executing a simple harmonic motion whose period of oscillation is given by

$$T = \frac{2\pi}{\omega} \text{ or } T = 2\pi\sqrt{\frac{I}{mgl}} \qquad ...(3)$$

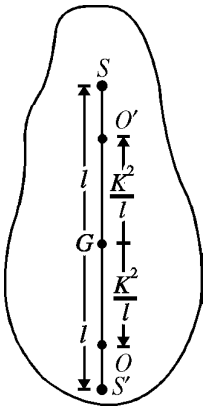


Fig. 15.

If I_g be the moment of inertia of the rigid body about a parallel axis through its centre of gravity *G*, then by the application of the theorem of parallel axis, we have

$$I = I_g + ml^2 = mK^2 + ml^2 = m(K^2 + l^2) \qquad ...(4)$$

where *K* is the radius of gyration of the body about a parallel axis through *G*.

Hence the expression of periodic time of the pendulum becomes

$$T = 2\pi\sqrt{\frac{m(K^2+l^2)}{mgl}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{\frac{K^2}{l}+l}{g}} \qquad ...(5)$$

Here, the periodic time is the same as that of a simple pendulum of length $\left(\frac{K^2}{l}+l\right)$, say L. This length L is called the **length of an equivalent simple pendulum.** If we take a point O on the line SG produced such that $SO = \frac{K^2}{l}+l$, the length of an equivalent simple pendulum, then the point O is called **centre of oscillation. (fig. 15)**

Interchangeability of centres of suspension and oscillation: The distance of the centre of oscillation from the centre of gravity $= K^2/l = l'$ (say).

Thus, the expression for the time period of the compound pendulum can be written as

$$T = 2\pi\sqrt{\frac{l'+l}{g}} \qquad ...(6)$$

If the pendulum is inverted and made to oscillate about the centre of oscillation, then its time period T' will be given by

$$T' = 2\pi\sqrt{\frac{(K^2/l')+l'}{g}}$$

But $K^2/l = l'$ or $K^2 = ll'$

$$\therefore \quad T' = 2\pi\sqrt{\frac{(ll/l')+l'}{g}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{l+l'}{g}} \qquad ...(7)$$

From eqs. (6) and (7), we get

$$T = T' \qquad ...(8)$$

Hence, *the time periods of oscillation about the centre of suspension and the centre of oscillation are equal, i.e., they are interchangeable.*

Condition for maximum and minimum time periods *(Variation of T with l)*: The period of oscillation of a compound pendulum, is given by

$$T = 2\pi\sqrt{\frac{(K^2/l)+l}{g}} \qquad ...(9)$$

Obviously, as l is varied, time period T also alters, if other factors remain constant. Relation (16) shows **the variation of time period with length.**

Eq. (9) can be written as

$$T^2 = \frac{4\pi^2}{g}\left(\frac{K^2}{l}+l\right).$$

Differentiating it with respect to l, we get

$$2T\frac{dT}{dl} = \frac{4\pi^2}{g}\left(-\frac{K^2}{l^2}+1\right). \qquad ...(10)$$

For minimum and maximum periods $dT/dl = 0$, i.e., $-K^2/l^2 + 1 = 0$ or $l = \pm K$

If we calculate the value of d^2T^2/dl^2 and put $l = K$, the quantity d^2T^2/dl^2 comes out to be positive. Hence for this value of l, time period of compound pendulum has minimum value. Condition for the period to be **minimum** is

$$l = K \qquad ...(11)$$

Thus, *the period of oscillation of the pendulum is minimum, when the distance between the centre of suspension and the centre of gravity is equal to the radius of gyration of the pendulum about a parallel axis through the centre of gravity.*

Now, if we put $l = 0$ in eq. (9), the value of T becomes infinite, i.e.,

$$T = \text{a maximum} \qquad ...(12)$$

In other words, *the period of oscillation of the compound pendulum is maximum, when the axis of suspension passes through centre of gravity of the pendulum.*

Four points, collinear with the C.G., about which the time period is the same: The time period of a compound pendulum, is given by

$$T = 2\pi\sqrt{\frac{K^2 + l^2}{lg}} \qquad ...(13)$$

where K is the radius of gyration of the compound pendulum about an axis passing through its centre of gravity and perpendicular to the plane of the oscillation and l the distance between the centre of suspension and its centre of gravity.

Squaring eq. (13), we get

$$T^2 = 4\pi^2\left(\frac{K^2 + l^2}{lg}\right) \quad \text{or} \quad l^2 + K^2 = \frac{lgT^2}{4\pi^2} \quad \text{or} \quad l^2 - \frac{gT^2}{4\pi^2}\cdot l + K^2 = 0 \qquad ...(14)$$

This is a quadratic equation in l and gives two values of l. If l_1 and l_2 be two values, then

$$l_1 + l_2 = \frac{gT^2}{4\pi^2} \qquad ...(15)$$

and

$$l_1 l_2 = K^2. \qquad ...(16)$$

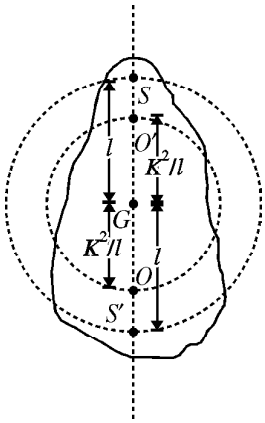


Fig. 16 : Compound Pendulum.

Both l_1 and l_2 are positive, because their sum and also product both are positive. From this we conclude that for a particular value of T there are two points distant l_1 and l_2 from the centre of gravity of the pendulum, about which the pendulum may be suspended and yet we have the same time period.

If $l_1 = l$, then $l_2 = K^2/l$ [from eq. (16)]

Now, if two circles are drawn with centre G and radii l and K^2/l **[fig. 16]**, the period of oscillation about any point of suspension on either circle is the same. Hence there are infinite number of points in a compound pendulum for which the time period has the same value. If a straight line through G is drawn, it will cut these circle in four points S, O', O, S'. Hence *in all there are four points collinear with the C. G. of the pendulum, about which the time periods are equal.*

Obviously, if S is the centre of suspension, O is the centre of oscllation or vice-versa; similarly if S' is the centre of suspension, $O'$is the centre of oscillation or vice-versa. Consider a pair of such points which (a) lie on a straight line passing through G, (b) lie on opposite side of G, and (c) lie at unequal distances from G. In **fig. 16**, S and O or S' and O' are such points, then

$$SO = SG + GO = l + \frac{K^2}{l} = \frac{l^2 + K^2}{l}.$$

Similarly, $$S'O' = \frac{l^2 + K^2}{l}.$$

Substituting in eq. (13), we get

$$T = 2\pi\sqrt{\frac{SO}{g}} = 2\pi\sqrt{\frac{S'O'}{g}} = 2\pi\sqrt{\frac{L}{g}}. \qquad ...(24)$$

where $SO = S'O' = L$ is the *length of an equivalent simple pendulum.*

Determination of the Value of *g* by Bar Pendulum

One of the convenient and simple form of a compound pendulum, used to determine the value of

g, is the bar pendulum. It is just a uniform metal bar *PQ* about one metre long, having holes drilled at equal distances along its length symmetrically about its centre of gravity **[fig. 17]**. A knife edge can be inserted into any desired hole and is kept on a horizontal support. Now the bar can be made to oscillate in a vertical plane. In an actual experiment, the bar is allowed to oscillate about the horizontal knife edge which is inserted in each hole in turn, starting from *P* to *G*. The period of oscillation corresponding to each hole is obtained by noting the time for twenty five oscillations. The bar is then inverted and similarly the periods are obtained about different holes from *G* to *Q*.

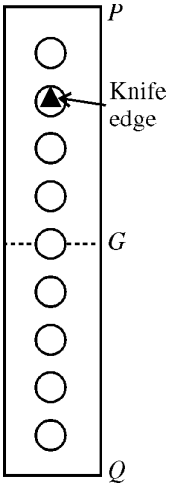


Fig. 17: Bar pendulum.

Now if a graph is plotted between the distance of the centre of suspension from one end of the bar and periodic time, it consists of two curves, symmetrical about a line which cuts *X*-axis at *G′* **[fig. 18]**. This point corresponds to the centre of gravity *G* of the bar about which the period is infinite. For a particular value of period *T*, if we draw a straight line parallel to *X*-axis, it cuts the two curves at the points *A*, *B*, *C* and *D*. Obviously if *A* is the centre of suspension, then *C* is the corresponding point of oscillation and *AC* is the length ($L = l + \frac{K^2}{l}$ = $AG'' + G''C$) of equivalent simple pendulum. Similar is the case for the points *B* and *D* and length *BD*. The mean of the distances *AC* and *BD* gives the length *L* of equivalent simple pendulum. Now one can use the relation $T = 2\pi\sqrt{\frac{L}{g}}$ to determine the value of *g*.

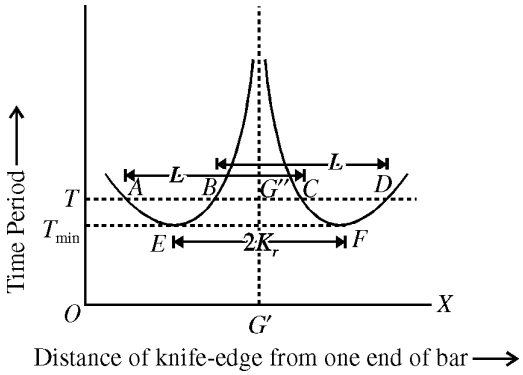


Fig. 18: Graph showing the variation of time periodwith distance for a bar pendulum.

Points of superiority of a compound pendulum over a simple pendulum: (1) A simple pendulum is an ideal concept only and cannot be realized in actual practice. In the case of a compound pendulum the length of an equivalent simple pendulum can be found easily and accurately. Thus the value of *g* can be determined accurately with compound pendulum.

(2) The distance between the two positions of knife edges can be measured accurately. In the case of a simple pendulum both the point of suspension and point of oscillation are indefinite and hence the true length of the pendulum can hardly be measured correctly.

(3) The compound pendulum vibrates as a whole and thus there is no possibility of a lag as it may present between the bob and the thread in the case of a simple pendulum.

(4) Due to its large mass and moment of inertia, the compound pendulum keeps on vibrating for a fairly long time and thus the time period can be measured accurately by finding the time for a large number of vibrations. In a simple pendulum the vibrations die out quickly as the mass of the bob is comparatively very small.

(b) Torsional Pendulum

If one end of a fairly thin and long wire is clamped to a rigid support and the other end is attached to the centre of a heavy body (e.g., disc, cylinder or sphere), then this arrangement is called the **torsional pendulum [fig. 19]**.

If the disc or sphere be turned in the horizontal plane to twist the wire and then released, it executes torsional vibrations of a definite period about the wire as axis.

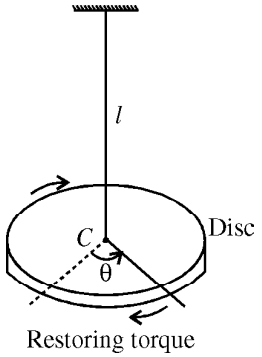


Fig. 19: Torsional pendulum : Restoring torque is opposite to θ increasing.

Now, if the disc is rotated through an angle θ, then the restoring torque, set up in the wire, will be $-C\theta$, where C ($= \eta\pi r^4/2l$) is the torsional rigidity or couple per unit twist of the wire. Here η is the modulus of rigidity of the material of the wire and the wire has length l with radius r.

This torque produces an angular acceleration $d^2\theta/dt^2$ in the disc or sphere. If I be the moment of inertia of the disc ($I = \frac{1}{2}MR^2$) or sphere ($I = \frac{2}{5}MR^2$) about the wire as axis, then the torque is $I\frac{d^2\theta}{dt^2}$. Hence the equation of motion is

$$I\frac{d^2\theta}{dt^2} + C\theta = 0 \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$$

Putting $\omega^2 = C/I$, we obtain

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad ...(1)$$

This equation represents a simple harmonic motion of period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{C}}. \qquad ...(2)$$

It may be noted that in the derivation of this formula no approximation is required. Hence the period of torsional oscillations remains the same for large amplitude oscillations, provided that the elastic limit of the suspension wire has not been exceeded.

The solution of eq. (1) is of the form

$$\theta = \theta_0 \sin(\omega t + \phi)$$

where θ_0 = angular amplitude, $\omega = \sqrt{C/I}$ and ϕ = initial phase.

Measurement of Rigidity Modulus (η) by Torsional Pendulum

We know that the expression for the torsional rigidity (C) of the wire used in the torsional pendulum, is

$$C = \frac{\eta\pi r^4}{2l} \qquad ...(3)$$

where η is the rigidity modulus of the material of the wire, r is the radius of the wire and l is its length.

Also the expression for moment of inertial (I) for a disc or cylinder about the axis of symmetry is given by

$$I = \frac{1}{2}MR^2 \qquad ...(4)$$

where M is the mass and R is the radius of the disc or cylinder.

In case of sphere, $$I = \frac{2}{5}MR^2 \qquad ...(5)$$

where M is the mass and R is the radius of the sphere.

In order to determine the value of rigidity modulus (η) of the wire by a torsional pendulum, the apparatus consists of a long thin specimen wire, whose one end in attached to a rigid support. To the other free end of the wire, a heavy disc (or cylinder or sphere) is fixed at its centre **(fig. 19)**. The disc is turned in the horizontal plane and then released, it executes torsional oscillations, whose period is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \qquad ...(6)$$

Substituting for I and C from eqs. (3) and (4) we obtain

$$T = 2\pi\sqrt{\frac{MR^2}{2}\cdot\frac{2l}{\eta\pi r^4}} = 2\pi\sqrt{\frac{MR^2 l}{\eta\pi r^4}}$$

whence $$\eta = \frac{4\pi MR^2 l}{T^2 r^4} \quad ...(7)$$

Periodic time T is obtained by measuring the time for 20 oscillations of the torsional pendulum. Now, substituting the values of M (mass) and R (radius) of the disc (or cylinder) and also the values of l (length) and r (radius) of the wire in eq. (7), we can determine the value of rigidity modulus of the material of the wire.

Ex. 1. *A disc of metal of radius R with its plane vertical can be made to swing about a horizontal axis passing through any one of a series of holes bored along a diameter. Show that the minimum period of oscillation, is given bys*

$$T = 2\pi\sqrt{\frac{1.414\,R}{g}}\,.$$

Sol. The vibrating disc will act as compound pendulum, whose time period is given by

$$T = 2\pi\sqrt{\frac{(K^2/l)+l}{g}}$$

where K is the radius of gyration of the disc and l the distance between the centre of gravity and the centre of suspension.

The condition for the time period to be minimum is $l = K$.

$\therefore$ $$T = 2\pi\sqrt{\frac{2K}{g}}$$

For disc, $\frac{1}{2}MR^2 = MK^2 \quad \therefore \quad K = \frac{R}{\sqrt{2}}$, where R is the radius of the disc.

$\therefore$ $$T = 2\pi\sqrt{\frac{2\times R}{\sqrt{2}g}} = 2\pi\sqrt{\frac{1.414\,R}{g}}\,.$$

This is the required expression.

Ex. 2. *A uniform thin rod of length* 120 *cm and width* 6 *cm is swinging in a vertical plane as pendulum about a point A at some distance from one end. If the time of swing is minimum, find the distance of A from the end of the rod.*

Sol. Moment of inertia of a uniform thin rod of length l and width b about the axis passing through C.G. and perpendicular to the plane of vibration is given by

$$I = M\left(\frac{l^2+b^2}{12}\right) = MK^2, \text{ where } K \text{ is the radius of gyration.}$$

Here, l = 120 cm = 1.2 m; b = 6 cm = 0.06 m.

$\therefore$ $$K^2 = \frac{l^2+b^2}{12} = 0.1203 \text{ m}^2.$$

If the time of swing is minimum, then distance between point of suspension and C.G. = K = $\sqrt{0.1203}$ m = 0.347 m.

Therefore, distance from one end of the rod = 0.60 – 0.347 = **0.253 m.**

Ex. 3. *A body of mass* 200 *gm oscillates about a horizontal axis at a distance of* 20 *cm from its centre of gravity. If the length of the equivalent simple pendulum be* 35 *cm find its moment of inertia about the axis of suspension.*

Sol. The oscillating body acts as compound pendulum, whose time period is given by

$$T = 2\pi\sqrt{\frac{(K^2/l)+l}{g}}.$$

The length $\frac{K^2}{l}+l = L$ is the length of the equivalent simple pendulum.

Here, l = 0.2 m, L = 0.35 m, $\therefore$ $0.35 = \frac{K^2}{0.20}+0.2$, i.e., $K^2 = 0.03$.

The moment of inertia I of the body about the axis of suspension is

$I = mK^2 + ml^2 = m(K^2 + l^2) = 0.2\ (0.03 + 0.2^2)$ = **0.014 kg-m^2.**

Ex. 4. *A circular disc of radius* 20 *cm oscillates as a pendulum about a point on its circumference. Calculate the period of oscillation.*

Sol. Let the disc of radius 20 cm be suspended from the point P of its circumference. **(fig. 20)**

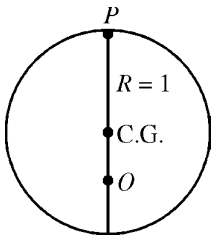


Fig. 20.

The moment of inertia of the disc, about an axis passing through its C.G. and perpendicular to its plane will be given by

$$I = MK^2 = \frac{1}{2}M\times(0.2)^2 \text{ or } K^2 = 0.02$$

The distance of centre of oscillation from the point of suspension is

$$L = PO = (K^2/l) + l = 0.3 \text{ m}.$$

$\therefore$ Period of oscillation, $T = 2\pi\sqrt{\frac{L}{g}} = 2 \times 3.14 \times \sqrt{\frac{0.3}{9.8}}$ = **1.098 sec.**

Ex. 5. *A compound pendulum is formed by suspending a heavy ring of radius* 490.5 *cm from a string. What is the length of the string when the period is minimum*? *Find this period.*

Sol. The time period of oscillation of a compound pendulum, is given by

$$T = 2\pi\sqrt{\frac{(K^2/l)+l}{g}} \qquad ...(i)$$

Here, a heavy ring is suspended with the help of a string. If the weight of the string is negligible with respect to the heavy ring, C.G. of the system will lie at the centre of the ring.

From eq. (i) the time period will be minimum, when $K = l$

But for the ring, $I = MR^2 = MK^2$ or $R = K$.

where R is the radius of the ring (consider all the mass is concentrated at the rim).

$\therefore$ $R = l$

i.e., the distance between centre of suspension and centre of ring is R. In other words, for minimum time period, the length of the string must be zero and ring must be suspended to any point of the circumference. Putting in (i) $K = l = R$, we get

$$T_{\min} = \sqrt{\frac{(l^2/l)+l}{g}} = 2\pi\sqrt{\frac{2l}{g}} = 2\pi\sqrt{\frac{2R}{g}}.$$

Here, R = 490.5 cm and g = 981 cm/sec^2.

$\therefore$ $T_{\min} = 2 \times 3.14 \times \sqrt{\frac{2\times490.5}{981}}$ = **6.28 sec.**

Ex. 6. *A thin uniform bar of length* 120 *cm is made to oscillate about an axis through its end. Find the period of oscillation and other points about which it can oscillate with the same period.*

Sol. The moment of inertia of a thin bar of length a and mass M, is given by

$$l = \frac{Ma^2}{12} = MK^2, \text{ where } K \text{ is the radius of gyration.}$$

$$\therefore \qquad K^2 = \frac{a^2}{12} = \frac{120\times120}{12} = 1200.$$

The time period of compound pendulum, is given by

$$T = 2\pi\sqrt{\frac{(K^2/l)+l}{g}} = 2\pi\sqrt{\frac{(1200/60)+60}{980}} \qquad [\because\ l = SG = 60 \text{ cm}]$$

$$= 2 \times 3.14 \times \frac{2}{7} = \mathbf{1.796\ sec.}$$

Let L be the length of equivalent simple pendulum, then

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$\therefore \qquad L = \frac{gT^2}{4\pi^2} = \frac{980}{(3.14)^2}\times\frac{(2\times3.14\times2)^2}{49} = 80 \text{ cm} = \mathbf{0.8\ m.}$$

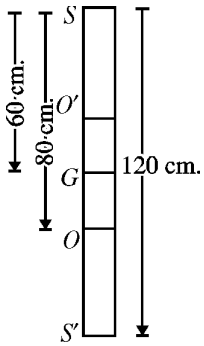


Fig. 21

Therefore, there is another point O at a distance of 80 cm, from S or 20 cm from C.G. about which time period will be the same.

If the pendulum is inverted and suspended from S', time period will be the same due to symmetry. Corresponding to S', there will be a point O at a distance of 20 cm from C.G.

Hence the bar will have same time periods about four points S, S', O, O' whose distance from S are 0, 120, 80 and 40 cm respectively **[fig. 21]**.

Ex. 7. *A soild sphere of mass* 2 *kg and radius* 0.05 *metre is suspended from a wire. Find the period of oscillation, if the torque required to twist the wire is* 4×10^{-3} *newton-m/radian.*

Sol. Moment of inertia of the sphere about any diameter is

$$I = \frac{2}{3}MR^2 = \frac{2}{3}\times2\times0.05\times0.05 = 0.002 \text{ kg} \times \text{m}^2.$$

$\therefore$ Period of oscillation, $T = 2\pi\sqrt{\frac{I}{C}} = 2\pi\times\sqrt{\frac{0.002}{4\times10^{-3}}} = \mathbf{4{\cdot}44\ sec.}$

Ex. 8. *A body is executing torsional oscillations about a wire of torsional rigidity C. If the total energy of oscillation is E and the moment of inertia of the body about the axis of rotation is I, show that the angular velocity* $(d\theta/dt)$ *at the displacement* θ *is given by*

$$\frac{d\theta}{dt} = 2\pi\sqrt{\frac{C}{L}}\sqrt{\frac{2E}{C}-\theta}\,.$$

Sol. Let C be the couple per unit twist. If the wire is twisted through an angle θ, then the restoring couple is $-C\theta$. Now if the wire is twisted to angle $\theta + d\theta$, then the amount of work done is

$$dW = C\theta d\theta$$

Hence the total amount of work done in twisting the wire through an angle θ is

$$W = \int_0^\theta C\theta\, d\theta = \frac{1}{2}C\theta^2$$

At the angular displacement θ, this work done becomes the potential energy of the system *i.e.,*

$$U = \frac{1}{2}C\theta^2\,.$$

The instantaneous angular displacement θ of a body, executing torsional oscillation is given by

$$\theta = \theta_0 \sin(\omega t + \phi), \text{ where } \omega = \sqrt{\frac{C}{I}}$$

So that $\frac{d\theta}{dt} = \theta_0\omega \cos(\omega t + \phi)$*

K.E. at the angular displacement θ is

$$K = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2}I\theta_0{}^2\omega^2\cos^2(\omega t+\phi)$$

$$= \frac{1}{2}I\theta_0{}^2\omega^2\left(1-\frac{\theta^2}{\theta_0{}^2}\right) = \frac{1}{2}I\omega^2(\theta_0{}^2-\theta^2) = \frac{1}{2}C(\theta_0{}^2-\theta^2)$$

$\therefore$ Total energy, $E = K + U = \frac{1}{2}C(\theta_0{}^2-\theta^2)+\frac{1}{2}C\theta^2 = \frac{1}{2}C\theta_0{}^2$

$\therefore$ Angular amplitude $\theta_0 = \sqrt{\frac{2E}{C}}$

Now, $\frac{d\theta}{dt} = \theta_0\sqrt{1-\frac{\theta^2}{\theta_0{}^2}} = \sqrt{\frac{C}{I}}\sqrt{\theta_0{}^2-\theta^2} = \sqrt{\frac{C}{I}}\sqrt{\frac{2E}{C}-\theta^2}$.

Ex. 9. *A sphere of moment of inertia* 3355 *gm-cm*^{2} *when suspended by a thin wire executes torsional oscillations of period* 4.3 *seconds. Calculate the maximum angular velocity of the sphere and its average potential energy when the amplitude is* π/2 *radians.*

Sol. Let the equation for the angular displacement of the sphere be

$$\theta = \theta_0 \sin(\omega t + \phi)$$

$\therefore$ Angular velocity $d\theta/dt = \theta_0\omega \cos(\omega t + \phi)$

Its maximum value = $\theta_0\omega = 2\pi n\theta_0$.

Here $\theta_0 = \pi/2$ radians and frequency $n = 1/4.3$ per sec.

$\therefore$ Max. angular velocity = $2 \times 3.14 \times \frac{1}{4.3}\times\frac{3.14}{2}$ = **2.3 rad/sec.**

Average potential energy = $\frac{1}{4}I\omega^2\theta_0{}^2$ [In place of $\frac{1}{2}m\omega^2a^2$]

$$= \frac{1}{4}\times 3355\times 10^{-7}\times\left(\frac{2\pi}{4.3}\right)^2\times\left(\frac{\pi}{2}\right)^2 = \mathbf{4.45 \times 10^{-4}\ J.}$$

7. Additional Examples of Harmonic Oscillator

(a) Simple Pendulum

A simple pendulum consists of a small heavy (ideally point) bob of mass *m*, suspended by means of an inextensible massless string of length *l* to a rigid support *S* **[fig. 22]**. If the bob is drawn to one side from its equilibrium position *O* and then left free, it starts to oscillate in a vertical plane. Let θ be the angular displacement at any time *t*. Due to the weight *mg* of the bob, a restoring torque (i.e., moment of force) τ is acting on the system and is given by $\tau = -mgl \sin\theta$.

This is the trorque about the point *S* and bears negative sign because its direction is opposite to θ increasing. If *I* be the moment of inertia of the bob about the point *S* and $d^2\theta/dt^2$ be the angular acceleration at the angular displacement θ, then torque can be expressed as $\tau = I\frac{d^2\theta}{dt^2}$

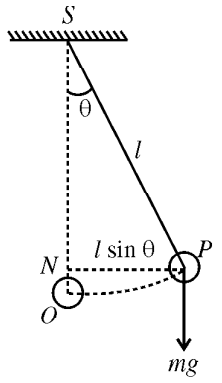


Fig. 22: Simple pendulum.

* Here, ω is not the actual angular velocity but governing the phase of motion.

Therefore, $$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta \quad ...(1)$$

If the bob is considered a point mass m, then its moment of inertia about S is given by

$$I = ml^2 \quad ...(2)$$

Hence $$ml^2\frac{d^2\theta}{dt^2} = -mgl\sin\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

or $$\frac{d^2\theta}{dt^2} + \omega^2\sin\theta = 0, \quad \text{where } \omega = \sqrt{\frac{g}{l}}, \quad ...(3)$$

Taylor's expansion of $\sin\theta$ is

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - ..... \quad ...(4)$$

For sufficiently small θ (in radians), we can neglect all terms in the expansion except the first term. In such a case,

$$\sin\theta \approx \theta \quad ...(5)$$

Thus for small angular amplitude, eq. (3) is obtained to be

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad ...(6)$$

Eq. (6) is completely analogous to the differential equation of the harmonic oscillator. The solution of the equation is given by

$$\theta = \theta_m \sin(\omega t + \theta) \quad ...(7)$$

where θ_m is the maximum angular displacement and ϕ is the phase constant.

The period of oscillation of the simple pendulum is given by

$$T = \frac{2\pi}{\omega} \quad \text{or} \quad T = 2\pi\sqrt{\frac{l}{g}} \quad ...(8)$$

(b) Charge Oscillations in *L*-*C* Circuit

Let us consider a coil of self-inductance L and negligible resistance connected in series with a capacitor of capacitance C [**fig. 23**]. Initially the condenser is charged by a battery to a charge $+Q_0$ on the right plate. Next the connection of the battery is cut off with the help of a key and the condenser is discharged through the inductance L.

Let Q be the charge at any time t on the capacitor C in the circuit. The potential difference across the capacitor C is $V = Q/C$. We take V to be positive when the right plate of the condenser is positively charged and represent the current direction by the arrow in **fig. 23**. Obviously the voltage across the capacitor is the same as the induced voltage $L\,di/dt$ across the inductance L, where di/dt is the rate of change of the current in the circuit. Thus

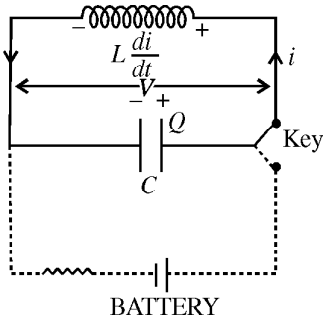


Fig. 23 : L. C. circuit.

$$V = \frac{Q}{C} = L\frac{di}{dt} \quad ...(1)$$

But $i = -dQ/dt$ (the charge is decreasing on the right plate and the current is flowing as shown),

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \quad ...(2)$$

Eq. (2) is analogous to the differential equation of simple harmonic motion of spring-mass system. We can think Q as the charge displacement, L as the charge inertia and $1/C$ as the force constant. Here the electromotive force (Q/C) across the capacitance plays the role of restoring force in the mechanical oscillator.

Eq. (2) can be written as

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0 \text{ or } \frac{d^2Q}{dt^2} + \omega^2 Q = 0 \quad ...(3)$$

This is identical to equation of the harmonic oscillator. Thus the charge on the condenser is oscillatory and the frequency of oscillation is given by

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \quad ...(4)$$

The solution of eq. (3) is expressed as

$$Q = Q_0 \sin(\omega t + \phi) \quad ...(5)$$

Thus we find that the charge on the condenser oscillates between the values $+ Q_0$ and $- Q_0$ and its frequency depends on L and C. Differentiating eq. (5) with respect to t, we get the instantaneous current i, i.e.,

$$i = - dQ/dt = - Q_0\omega \cos(\omega t + \phi)$$

or
$$i = i_0 \sin\left(\omega t + \phi - \frac{\pi}{2}\right) \quad ...(6)$$

where $Q_0\omega = i_0$.

Thus the current lags in phase by angle $\pi/2$ to the charge.

Practically there is always some resistance in L-C circuit and its effect will be considered later. Now let us calculate the energy stored in the inductance L and capacitance C at any instant t. The current is build up from 0 to i in time t and the energy E_L stored in the inductance L is obtained by integrating the instantaneous power with respect to time *i.e.*,

$$E_L = \int_0^t Vi\,dt = \int_0^t \frac{di}{dt} i\,dt = \int_0^I Li\,di = \frac{1}{2}Li^2$$

The potential energy stored at the instant t in the capacitor is

$$E_C = \frac{1}{2}\frac{Q^2}{C}$$

Therefore the total energy in the circuit at any time t is

$$E = E_L + E_C = \frac{1}{2}Li^2 + \frac{1}{2}\frac{Q^2}{C} \quad ...(7)$$

This equation for total energy is similar to the equation for mechanical oscialltor ($E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$). As Q and i vary with time, the inductance and capacitor exchange energy periodically similar to the energy exchanges in the spring and mass system.

(c) Oscillation of Two Particles Connected by a Spring-diatomic Molecule

Let us consider two masses m_1 and m_2 connected by a weightless spring. The masses are free to vibrate only along the axis of the spring. Such an oscillating system is called **two body harmonic oscillator.** When due to some reason one of the masses is slightly displaced from its equilibrium position, this produces an extension or compression in the spring. Due to spring action, it exerts a linear restoring force on the masses so that both of the masses vibrate about their mean equilibrium positions and the centre of mass of the system remains stationary.

Suppose that two masses m_1 and m_2 are connected by a massless spring of force constant k. The masses are constrained to move along the axis of the spring **[fig. 24]**. Let r_0 be the normal length of the

spring. Let x_1 and x_2 be the coordinates of the two ends of the spring at any instant t. Therefore the instantaneous length of the spring is $r = x_2 - x_1$. The change in length is thus obtained to be

$$x = r - r_0 = (x_2 - x_1) - r_0 \quad ...(1)$$

For $x > 0$, $x = 0$ and $x < 0$, the spring is extended, normal and compressed respectively. Suppose that at any instant, the spring is increased in length i.e., $x > 0$. The spring is exerting the same force (kx) in magnitude on the two masses but the force F_1 (= kx), acting on m_1, is opposite to the force F_2 (= $-kx$) on m_2 i.e.,

$$F_1 = kx \quad \text{and} \quad F_2 = -kx \quad ...(2)$$

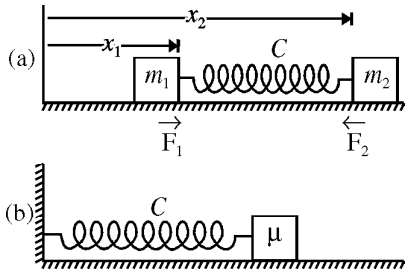


Fig. 24: (a) Two-body oscillator: $F_2 = -C_x = -F_1$, $x = (x_2 - x_1) - r_0$, r_0 = unstretched length of the spring of force constant C; (b) Equivalent single-body oscillator of mass $\mu\left(=\frac{m_1 m_2}{m_1 + m_2}\right)$ attached to a rigid wall by a spring of the same force constant C.

Applying Newton's second law, we obtain the equations of motion of the two masses as follows:

$$m_1 \frac{d^2 x_1}{dt^2} = kx \quad \text{and} \quad m_2 \frac{d^2 x_2}{dt^2} = -kx$$

$$\therefore \quad \frac{d^2 x_1}{dt^2} = \frac{k}{m_1} x \quad \text{and} \quad \frac{d^2 x_2}{dt^2} = -\frac{k}{m_2} x$$

On subtraction

$$\frac{d^2 x_2}{dt^2} - \frac{d^2 x_1}{dt^2} = -\frac{k}{m_2} x - \frac{k}{m_1} x \quad \text{or} \quad \frac{d^2 (x_2 - x_1)}{dt^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right) kx$$

or $$\frac{d^2 x}{dt^2} = -\frac{k}{\mu} x \quad \text{or} \quad \frac{d^2 x}{dt^2} + \frac{k}{\mu} x = 0 \quad ...(3)$$

where $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ or $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the **reduced mass** of the system. We have used $\frac{d^2 (x_2 - x_1)}{dt^2} = \frac{d^2 x}{dt^2}$ because r_0 is constant.

Eq. (3) is identical in form to the equation for simple harmonic oscillator, developed for one-body oscillations. The frequency of oscillation of the system is given by

$$\nu = \frac{1}{2\pi} \sqrt{\frac{k}{\mu}} \quad ...(4)$$

Thus the system has the same frequency of oscillation as a single object of mass μ, connected by a similar spring of force constant k to a rigid wall. Hence the two-body oscillator **[fig. 24(a)]** is equivalent to one-body oscillator **[fig. 24(b)]**. However in **[fig. 24(a)]** x is the relative displacement of the two masses from their equilibrium positions rather than the displacement of a single mass from its equilibrium position and μ is the reduced mass tather than the mass of single object. The potential energy of the two-body harmonic oscillator, when the length of the spring changes by x, is given by

$$U = \int_0^x kx\, dx = \tfrac{1}{2} kx^2 = \tfrac{1}{2} k (r - r_0)^2 \quad ...(5)$$

Vibrations of a diatomic molecule: A diatomic molecule, *e.g.*, HCl molecule, is an example of two-body system which can oscillate along the axis of symmetry. The atoms in a diatomic molecule are coupled through electromagnetic forces and the bonding between them may be imagined like a spring. Thus we may consider a diatomic molecule as a system of two masses connected by a spring.

In a diatomic molecule, the binding forces are electromagnetic in origin which are composed of attractive and repulsive parts. The equilibrium separation r_0 between the two atoms of the molecule is that at which the attractive and repulsive forces exactly balance each other. If the separation is slightly increased, the two atoms try to come to the equilibrium distance under resultant attractive force (restoring

force) and if the separation is slightly decreased, they repel each other to have equilibrium configuration. Thus the vibrations of the molecule may take place, if the system is disturbed by giving some energy.

Ex. 1. *A simple pendulum possesses* 0.05 *joule energy when its length is* 2 *m and the amplitude of motion of the bob is* 4 *cm. Calculate its energy* (1) *when the length remains the same but the amplitude is doubled,* (2) *the amplitude remains the same but the length is made one-half.*

Sol. $E = \frac{1}{2}m\omega^2 a^2$; Here, E = 0.05 J, $\omega^2 = g/l = g/2$ and $a = 4 \times 10^{-2}$ m

$$\therefore \qquad 0.5 = \frac{1}{2}m(g/2)(4\times10^{-2})^2$$

(1) When $a' = 2 \times 4 \times 10^{-2}$ m, $E' = \frac{1}{2}m(g/2)(2\times4\times10^{-2})^2$

$$\therefore \qquad \frac{E'}{0.05} = 4 \quad \text{or} \quad E' = \mathbf{0.2\ J}$$

(2) Here $l' = 2/2 = 1$ m,

$$\therefore \qquad E'' = \frac{1}{2}m(g/2)(4\times10^{-2})^2$$

$$\therefore \qquad \frac{E''}{0.05} = 2 \quad \text{or} \quad E'' = \mathbf{0.1\ J}$$

Ex. 2. *What is the frequency of electrical oscillations in an inductance of* 10 *millihenry in series with a* 2 *μf capacitance. If the maximum potential difference across the condenser is* 5 *volts, find the energy of the oscillating system.* **(Rewa 2004; Bhopal 04)**

Sol. $$\upsilon = \frac{1}{2\pi\sqrt{LC}}$$

Here $L = 10 \times 10^{-3}$ henry and $C = 2 \times 10^{-6}$ farad.

$$\therefore \qquad \upsilon = \frac{1}{\sqrt{10\times10^{-3}\times2\times10^{-6}}} = \mathbf{1.1 \times 10^3\ Hz.}$$

$$\text{Energy, } E = \frac{1}{2}CV^2 = \frac{1}{2}\times2\times10^{-6}\times5^2 = \mathbf{25 \times 10^{-6}\ joules.}$$

Ex. 3. *Show that the total energy in the L–C circuit, remains constant with time. What is the expression obtained for the total energy*?

Sol. $$E = \frac{1}{2}Li^2 + \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}LQ_0{}^2\omega^2\cos^2(\omega t + \phi) + \frac{1}{2}\cdot\frac{Q_0{}^2}{C}\sin^2(\omega t + \phi)$$

But $$\omega^2 = \frac{1}{LC}, \quad \therefore \quad E = \frac{1}{2}\frac{Q_0{}^2}{C}.$$

Hence the total energy remains constant with time. The expression for total energy is given above.

Ex. 4. *Two masses of 10 gm and* 90 *gm are connected by a spring of length* 10 *cm and force constant* 10^3 *N/m*^{2}, *calculate the frequency of oscillation.*

Sol. $$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

Here $k = 10^3$ *N/m* and $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{0.01\times0.09}{0.01+0.09} = 0.009\,kg$

$$\therefore \qquad \nu = \frac{1}{2\pi}\sqrt{\frac{10^3}{0.009}} = \mathbf{53\,Hz}.$$

Ex. 5. *A sodium chloride molecule has a natural vibrational frequency of* 1.14×10^{13} Hz. *Calculate inter-atomic force constant. The masses of sodium and chlorine are* 23 *a.m.u.* and 35 *a.m.u. respectively.* (1 *a.m.u.* 1.67×10^{-27} kg). **(O.U. 2007; A.N.U. March 2007)**

Sol. Frequency of vibration for a diatomic molecule is given by

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

whence for constant, $k = 4\pi^2\nu^2\mu$

Here, $\nu = 1.14 \times 10^{13}$, $\mu = \dfrac{m_1 m_2}{m_1 + m_2} = \dfrac{23\times1.67\times10^{27}\times35\times1.67\times10^{-27}}{23\times1.67\times10^{-27}+35\times1.67\times10^{-27}} = 23.18 \times 10^{-27}$ kg

$\therefore$ $k = 4 \times (3.14)^2 \times (1.14 \times 10^{13})^2 \times 23.18 \times 10^{-27} =$ **118.9 N/m.**

Ex. 5. Calculate the vibrational frequency of HCl molecule, given the force constant = 839.0 N/m. ($m_H = 1.67 \times 10^{-27}$ kg and $m_{Cl} = 35 \times 1.67 \times 10^{-27}$ kg)

Sol. $\nu = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{\mu}} = \dfrac{1}{2\pi}\sqrt{\dfrac{839.0\times36}{35\times1.67\times10^{-27}}} =$ **11.4×10^{13} Hz.**

where $\mu = \dfrac{m_H m_{Cl}}{m_H + m_{Cl}} = \dfrac{1.67\times10^{-27}\times35\times1.67\times10^{-27}}{1.67\times10^{-27}+35\times1.67\times10^{-27}} = \dfrac{35\times1.67\times10^{-2}}{36}$

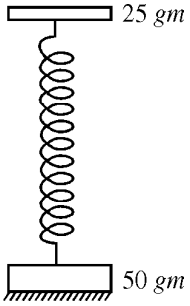


Fig. 25.

Ex. 6. *Consider two discs of masses* 25 *gm and* 50 *gm* (*as shown in* **fig. 25**) *with massless spring of force constant* 10^3 *N/m. Calculate the frequency of oscillation of the spring* (i) *when the system is resting on a table, and* (ii) *when the table is removed and the system is falling freely.*

Sol. In the first case, one disc is resting on the table and therefore the other disc of mass 25 gm will vibrate.

$\therefore$ Frequency, $\nu = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{10^3}{0.025}} = \dfrac{10^2}{\pi} =$ **31.8 Hz.**

In the second case, when the system is falling freely, both discs will bibrate and hence

$\nu = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{\mu}} = \dfrac{1}{2\pi}\sqrt{\dfrac{10^3\times3}{0.05}} =$ **39 Hz.** $\left(\because \mu = \dfrac{m_1 m_2}{m_1+m_2} = \dfrac{0.025\times0.05}{0.025+0.05} = \dfrac{0.05}{3}\text{kg}\right)$

8. Principle of Superposition

Let two (or more) simple harmonic motions be acting simultaneously on a particle (or point). At any instant, if x_1 be the displacement of the particle due to one simple harmonic motion and x_2 due to the other, then according to the principle of superposition, the resultant displacement of the particle is the vector sum of the two displacements, *i.e.*,

$$x = x_1 + x_2 \qquad ...(1)$$

We know that the differential equation for a simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad ...(2)$$

This is a **linear homogeneous equation** of second order. Such equation has the characteristic property that the *sum of any two solutions of it is itself a solution, i.e.,* if x_1 and x_2 are the solutions of eq. (2), then the sum $x = x_1 + x_2$ of these solutions also satisfies eq. (2). Thus, if x_1 and x_2 are the solutions of eq. (2), then

$$\frac{d^2x_1}{dt^2} = \omega^2 x_1 \quad \text{and} \quad \frac{d^2x_2}{dt^2} = \omega^2 x_2 \qquad ...(3)$$

The addition of the two equations gives

$$\frac{d^2(x_1+x_2)}{dt^2} = \omega^2(x_1 + x_2) \quad \text{or} \quad \frac{d^2x}{dt^2} = \omega^2 x \qquad ...(4)$$

This means that $x = x_1 + x_2$ satisfies eq. (2) and hence *the differential equation of simple harmonic motion is a linear homogeneous equation and follows the principle of superposition.* In the present work, we shall study only those oscillations which satisfy linear equations and are in accordance with the principle of superposition. It is to be mentioned that the differential equations of damped and driven harmonic oscillators are linear differential equations.

In case of nonlinear equation

$$\frac{d^2x}{dt^2} = - k_1x - k_2x^2 - k_3x^3 - ..... \qquad ...(5)$$

(e.g., for anharmonic oscillator), the superposition of two solutions does not satisfy the equation itself. In case of simple pendulum, the differential equation has the form

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0 \qquad [\because \omega = \sqrt{g/l}\,] \qquad ...(6)$$

But $$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} - \frac{\theta^5}{\lfloor 5} - .....$$

Hence $$\frac{d^2\theta}{dt^2} = - \omega^2\theta + \frac{\omega^2}{6}\theta^3 - \frac{\omega^2}{120}\theta^5 - .... \qquad ...(7)$$

Thus the differential equation for the motion of simple pendulum is nonlinear and in such a case, the principle of superposition will be not applicable. However for small θ, θ^3, θ^5 etc. terms can be neglected and then the equation.

$$\frac{d^2\theta}{dt^2} = - \omega^2\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad ...(8)$$

is the linear equation, respresenting simple harmonic motion, and for such small amplitude oscillations of simple pendulum, the principle of superposition is applicable. Large amplitude oscillations of simple pendulum are anharmonic and represented by nonlinear equation (7). In the present chapter, we shall apply the principle of superposition for compounding two simple harmonic motions which are in a straight line or mutually perpendicular. However this principle is applicable for the superposition of many linear simple harmonic motions.

9. Superposition of Two Simple Harmonic Motions of Same Frequency (or Equal Periods) along a Line

Let two simple harmonic motions of same frequency ($\upsilon = \omega/2\pi$) [equal periods ($= 2\pi/\omega$] but of different amplitudes and phases be simultaneously acted on a particle in the same straight line. Let the two component vibrations be represented by

$$x_1 = a_1 \sin(\omega t + \phi_1) \qquad ...(1)$$

and $$x_2 = a_2 \sin(\omega t + \phi_2) \qquad ...(2)$$

where a_1 and a_2 are the amplitudes and ϕ_1 and ϕ_2 the epochs or initial phases of the component vibrations. The phase difference between them is $(\phi_1 - \phi_2)$.

According to the principle of superposition, the resultant displacement of the particle is given by

$$x = x_1 + x_2 \qquad ...(3)$$

Thus,
$$\begin{aligned} x &= a_1 \sin(\omega t + \phi_1) + a_2 \sin(\omega t + \phi_2) \\ &= a_1[\sin\omega t \cos\phi_1 + \cos\omega t \sin\phi_1] + a_2[\sin\omega t \cos\phi_2 + \cos\omega t \sin\phi_2] \\ &= \sin\omega t[a_1\cos\phi_1 + a_2\cos\phi_2] + \cos\omega t[a_1\sin\phi_1 + a_2\sin\phi_2] \end{aligned} \qquad ...(4)$$

Now, suppose

$$a \cos \theta = a_1 \cos \phi_1 + a_2 \cos \phi_2 \quad ...(5)$$

and $$a \sin \theta = a_1 \sin \phi_1 + a_2 \sin \phi_2 \quad ...(6)$$

Then $$x = a \sin \omega t \cos \theta + a \cos \omega t \sin \theta \text{ or } x = a \sin (\omega t + \theta) \quad ...(7)$$

where a is the resultant amplitude of oscillation, $(\omega t + \theta)$ is its phase.

Eq. (7) represents that **the resultant motion is simple harmonic** whose frequency $(= \omega/2\pi)$ is the same as that of any component vibration, but the amplitude a and phase θ are different. Squaring and adding eqns. (5) and (6) we obtain

$$a^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos (\phi_1 - \phi_2)$$

or $$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos(\phi_1 - \phi_2)} \quad ...(8)$$

which tells us the **amplitude of resultant simple harmonic motion.**

Dividing eq. (6) by eq. (5),

$$\tan \theta = \frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2} \text{ or } \theta = \tan^{-1} \frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2} \quad ...(9)$$

This angle θ gives the **phase of the resultant motion.**

Let us consider some special cases:

(a) If $\phi_1 - \phi_2 = 0$ or $2n\pi$, where n is an integer, the two vibrations are in the **same phase** and the resultant amplitude has the maximum value, *i.e.,*

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2} = a_1 + a_2 \quad ...(10)$$

(b) If $\phi_1 - \phi_2 = \pi$ or $(2n + 1)\pi$, where n is any interger, the two vibrations are in the **opposite phase** and the resultant amplitude has the minimum value *i.e.,*

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2} = a_1 - a_2 \quad ...(11)$$

If in addition $a_1 = a_2$, the resultant amplitude $a = 0$, *i.e.,* the particle remains in rest position.

10. Combination of Two Mutually Perpendicular Simple Harmonic Vibrations of Same Frequency

Let the two simple harmonic motions of the same frequencies be acting along the axis of X and axis of Y respectively. If their displacements at any time t be x and y, then they can be represented by

$$x = a \sin (\omega t + \phi) \quad ...(1)$$

$$y = b \sin \omega t \quad ...(2)$$

where a and b are the amplitudes of the component vibrations having same frequency $\omega/2\pi$ (or period $2\pi/\omega$) and the first vibration is ahead of the second by a phase angle ϕ, *i.e.,* the phase difference between the two vibrations is ϕ.

From eq. (1), we have $$x = a[\sin \omega t \cos \phi + \cos \omega t \sin \phi] \quad ...(3)$$

But from eq. (2) $$\sin \omega t = \frac{y}{b} \text{ and } \cos \omega t = \sqrt{1 - \frac{y^2}{b^2}}$$

Substituting the values of $\sin \omega t$ and $\cos \omega t$ in (3), we get

$$\frac{x}{a} = \frac{y}{b} \cos \phi + \sin \phi \sqrt{1 - \frac{y^2}{b^2}} \text{ or } \left(\frac{x}{a} - \frac{y}{b} \cos \phi\right)^2 = \sin^2 \phi \left(1 - \frac{y^2}{b^2}\right)$$

Thus $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cos \phi = \sin^2 \phi \quad ...(4)$$

This expression represents the resultant motion, which is in general an ellipse inclined to the axes of co-ordinates.

Special Cases:

(i) When $\phi = 0$, the relation (4) becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}=0 \quad \text{or} \quad \left(\frac{x}{a}-\frac{y}{b}\right)^2=0 \text{ or } \pm\left(\frac{x}{a}-\frac{y}{b}\right)=0. \qquad ...(5)$$

This equation represents a pair of coincident straight line $y = bx/a$, passing through the origin and lying in the first and third quadrants [**fig. 26(a)**].

This motion along a straight line represents the case of *linearly polarized light* in optics.

(ii) When $\phi = \pi/4$, we have $\sin\phi = 1/\sqrt{2}$ and $\cos\phi = 1/\sqrt{2}$. Hence eq. (4) takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{\sqrt{2}xy}{ab}=\frac{1}{2} \qquad ...(6)$$

This equation represents an oblique ellipse **[fig. 26(b)]**.

(iii) *When* $\phi = \pi/2$, the expression (4) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ...(7)$$

This equation represents an ellipse, whose axes are coincident with the *X*-and *Y*-axes [**fig. 26(b)**]. The semi-major and semi-minor axes of the ellipse are *a* and *b*. Further as the time *t* starts to increase from zero, *x* begins to decrease from its maximum positive value and *y* simultaneously begins to go positive from $y = 0$ position. Thus the ellipse is described in the anticlockwise direction [**fig. 26(c)**]. The same ellipse is obtained for $\phi = 3\pi/2$ or $-\pi/2$, but now the motion is in clockwise direction. Thus when the phase difference ϕ between the two rectangular simple harmonic motions is $+\pi/2$, the resultant motion is *elliptically polarized.*

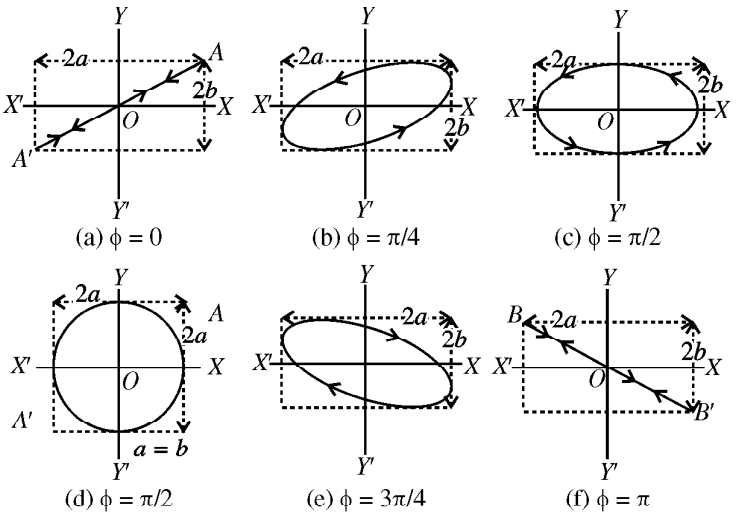


Fig. 26.

Now, if $a = b$, the ellipse degenerates in the circle **[fig. 26(d)]**

$$x^2 + y^2 = a^2 \qquad ...(8)$$

Hence, *two simple harmonic motions at right angles to each other of equal amplitudes but with phases differing by* $\pi/2$, *are equivalent to a uniform circular motion, the radius of the circle being equal to the amplitude of either simple harmonic motion.* This case corresponds to *circularly polarized light* in optics.

Conversely, *a uniform circular motion can be resolved into two simple harmonic motions at right angles to each other, their amplitudes being equal, but their phases differing* $\pi/2$.

(iv) When $\phi = 3\pi/4$, we have $\sin\phi = 1/\sqrt{2}$ and and $\cos\phi = -1/\sqrt{2}$.

Therefore, from eq. (4), we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{\sqrt{2}xy}{ab}=\frac{1}{2} \qquad ...(9)$$

This equation represents an oblique ellipse as shown in **fig. 26(e)**.

(v) *When* $\phi = \pi$, eq. (4) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\sqrt{2}xy}{ab} = 0 \quad \text{or} \quad \left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0 \qquad ...(10)$$

which represents a pair of coincident straight lines $y = -bx/a$, passing through the origin and lying in the second and fourth quadrants as BOB' [**fig. 26(f)**].

For any other value of ϕ, except discussed above, the path will be an ellipse, inclined to the co-ordinate axes.

11. Composition of Two Rectangular S.H.M.'s of Nearly Equal Frequencies

If the two component vibrations are exactly of equal frequencies (or periods), the elliptical paths, referred to above, remain perfectly steady. But, when the two frequencies differ slightly from each other, there comes about a gradual but progressive change in the relative phase (ϕ) of the two vibrations and consequently the shape of the ellipse slowly undergoes a change. All these elliptic path lie within a rectangle of sides $2a$ and $2b$.

Now, starting with the phase in agreement, *i.e.*, $\phi = 0$, the ellipse coincides with the diagonal $\frac{x}{a} - \frac{y}{b} = 0$ of the rectangle.

As ϕ increases from 0 to $\pi/2$, the ellipse opens out to the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, passing through intermediate oblique positions.

When ϕ increases from $\pi/2$ to π, the ellipse closes up again and finally coincides with the other diagonal $\frac{x}{a} + \frac{y}{b} = 0$ of the rectangle [**fig. 27**].

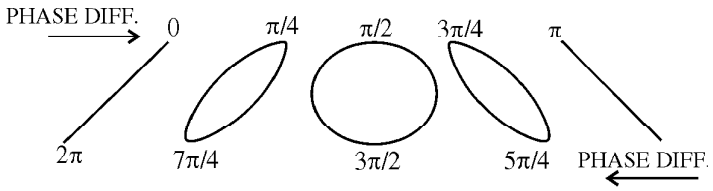


Fig. 27.

As the phase difference changes from π to 2π, the reverse process takes place, until the ellipse again coincides with the first diagonal. All these changes have been represented in **fig. 27**.

It should be noted that *the frequency of the complete cycle is equal to the difference of the frequencies of the two component simple harmonic vibrations.*

12. Composition of Two Rectangular S.H.M.'s of Frequencies in the Ratio 2 : 1

Let the two component rectangular simple harmonic vibrations be represented along X-axis and Y-axis as

$x = a \sin(2\omega t + \phi)$ ...(1)

and $y = b \sin \omega t$ ...(2)

where ϕ is the phase angle by which the first motion is ahead of the second. Clearly the frequency 4of (1) vibrations is double to that of (2).

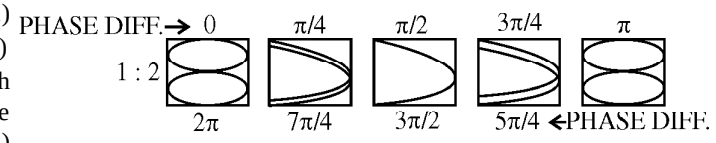


Fig. 28.

The resultant motion, which can be obtained by eleminating ωt from the equations (1) and (2), in general will be a curve, having two loops for any value of phase difference and amplitudes.

Here, we will consider only two special cases, given below:

(a) *When* $\phi = 0$, we get

$$x = a \sin 2\omega t = 2a \sin \omega t \cos \omega t$$

or $\frac{x}{a} = 2 \sin \omega t \cos \omega t$ or $\frac{x}{a} = \frac{2y}{b}\sqrt{1-\frac{y^2}{b^2}}$ or $\frac{x^2}{a^2} = \frac{4y^2}{b^2}\left(1-\frac{y^2}{b^2}\right) = -\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}-1\right)$

Thus, $$\frac{x^2}{a^2}+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}-1\right) = 0 \qquad ...(3)$$

This equation represents the figure of '8' [**fig. 28**].

(b) *When* $\phi = \pi/2$, we have

$$x = a \sin (2\omega t + \tfrac{\pi}{2}) = a \cos 2\omega t = a(1-2 \sin^2 \omega t)$$

$$\therefore \quad \frac{x}{a} = 1 - \frac{2y^2}{b^2} \qquad \left[\because \sin^2 \omega t = \frac{y^2}{b^2}\right]$$

or $$\frac{2y^2}{b^2} = 1 - \frac{x}{a} = -\frac{x-a}{a}, \quad \therefore \quad y^2 = -\frac{b^2}{2a}(x-a) \qquad ...(4)$$

This is the equation of a parabola with vertex $(a, 0)$.

If the frequencies of the two rectangular vibrations depart slightly from the ratio 2 : 1, the form of the curve slowly varies as ϕ changes [**fig. 28**].

13. Lissajou's Figures

When a particle oscillates simultaneously under the action of two simple harmonic motions at right angles to each other, the curve traced by the resultant motion of the particle is called **Lissajous figure** after the name of J.A. Lissajous who made an extensive study of such motions. The shape of the curve depends upon the amplitudes, frequencies and phase difference of the component vibrations. We have seen that if the frequencies of the two S.H.Ms. at right angles are equal, the Lissajous figure, in general, is an ellipse and has a definite shape for a particular phase difference [**fig. 26**]. For $\phi = 0$ or π, it is a straight line and for $\phi = \frac{\pi}{2}$ and $a_1 = a_2$, it is a circle.

When the two frequencies of component perpendicular vibrations are in the ratio 2 : 1, the Lissajous figures are relatively complex. It has the shape of figure '8' for $\phi = 0$ or π and that of parabola for $\phi = \frac{\pi}{2}$. If the component frequencies are in the complicated ratio, the Lissajous figures are odd and complicated.

Experimental Demonstration of Lissajous Figures

(1) Optical method: For this purpose, two electrically maintained tuning forks *A* and *B* are set up so that the vibrations of one take place in a vertical plane and those of the other in a horizontal plane. Each fork has a small mirror attached to the side of a prong. Light from a small powerful source, rendered convergent with the help of a convex lens, is allowed to fall on the mirror *M* of the vertical fork *A*. The light, reflected from M_1, strikes the mirror M_2 of the horizontal fork *B*, from where it is directed towards the screen *S*. The position of the lens is so adjusted that it produces a sharp image on the screens [**fig. 29**].

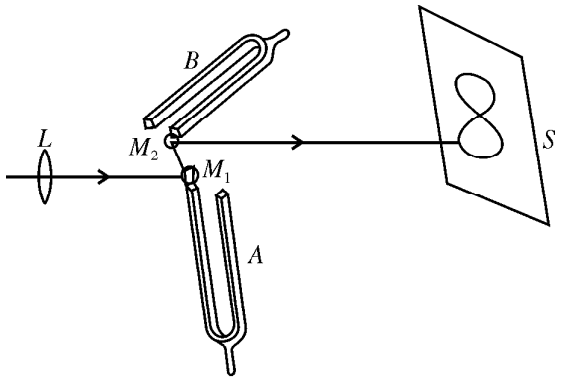


Fig. 29.

If now only the vertical fork is vibrated, the spot of light on the screen executes a vertical S.H.M. so rapidly, that only a bright vertical line is observed on the screen. If the horizontal fork alone vibrates, a bright horizontal line is seen. Thus these motions represent two simple harmonic vibrations at right angles to each other. Now, if

both the forks vibrate simultaneously, the two rectangular vibrations of the spot of light are compounded and the spot describes the path of the resultant motion on the screen. If the frequencies of the forks bear some simple ratio to each other, a Lissajous figure will be seen on the screen. For example, if the frequencies are in the ratio 2 : 1, a figure of '8' may be obtained. In case if the ratio between the frequencies is approximately two, it will gradually pass through the cyclic of changes of figures as shown in **fig. 29**. This method is usually employed for the comparison of the frequencies of two forks.

(2) By cathode ray oscillograph: Lissajous figures can be produced easily by means of a cathode ray oscilloscope (CRO). Different alternating sinusoidal voltages are applied at X and Y deflection plates of the CRO and the electron beam traces the resultant effect on the flourescent screen. When the applied voltages have the same frequency, we can obtain the various curves of **fig. 26**, by adjusting the phases and amplitudes. By using the CRO, we can also obtain easily the Lissejous figures for the component frequencies for the ratio 1 : 2 or any other ratio.

Uses of Lissajous Figures

(1) Lissajous figures are used in obtaining beautiful designs for printing in cloth industry.

(2) Lissajous figures find useful applications in the acoustical measurements, *e.g.*, determintaion of unknown frequency. If the frequencies of vibration of the two rectangular vibrations are not exactly equal or in the ratio 2 : 1, the form of the Lissajous figure undergoes a gradual progressive change. The time, taken by the curve to undergo a complete cycle of changes, enables us to find the frequency of one vibration, if the frequency of the other is known. Let the frequency of one vibration be n and that of the other be slightly different n', the frequencies being nearly in the ratio 1 : 1. If t be the time in which the cycle undergoes a complete change, then the frequency of the other will be $n' = n \pm 1/t$.

If the frequencies are nearly in the ratio 2 : 1, then $n' = 2n \pm 1/t$.

To find whether n' is greater of less than n or $2n$ (as the case may be) a small piece of wax is attached to the unknown (*e.g.*, the prong of the fork) and again the time of one cycle of changes is determined. If this time increases, the unknown frequency is higher and if it decreases, the unknown frequency is lower than n or $2n$.

(3) One practical application of Lissajous figures is that they enable us to know by inspection the period ratio of its constituent vibrations. It is evident that if the horizontal and vertical lines are drawn on a Lissajous figure and if they cut it m and n times respectively, the required period ratio of the two vibrations is $m : n$. *i.e.*,

$$\frac{\text{Frequency along } Y\text{-direction } (\nu_2)}{\text{Frequency along } X\text{-direction } (\nu_1)} = \frac{m}{n}$$

$$= \frac{\text{Number of intersection points of the Lissajous figure by a parallel line to } X\text{-axis}}{\text{Number of intesection points of the figure by a parallel line to } Y\text{-axis}}$$

Ex. 1. *A simple pendulum is made to oscillate under two simple harmonic motions, given by* $x = 5 \cos \omega t$ *and* $y = 6 \cos (\omega t + \phi)$.

Find the resultant path of the bob, when $\phi = 0$ *and* $\frac{\pi}{2}$. *Draw a plot of the path in each case, indicating the direction of movement of the bob.*

Sol. For $\phi = 0$, $x = 5 \cos \omega t$ and $y = 6 \cos \omega t$

$$\therefore \quad \frac{y}{x} = \frac{6}{5} \quad \text{or } y = 1.2x$$

In this case the path of the bob is straight line as shown in **fig. 30(a)**.

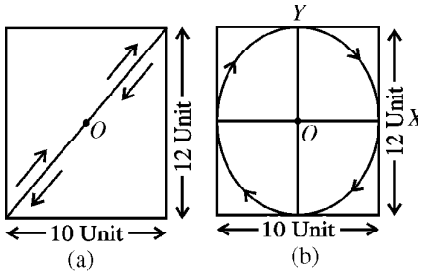


Fig. 30.

For $\phi = \frac{\pi}{2}$, $x = 6 \cos \omega t$

and $y = -6 \sin \omega t$

$\therefore \quad \sin^2 \omega t + \cos^2 \omega t = \frac{y^2}{6^2} + \frac{x^2}{5^2}$

or $\frac{x^2}{25} + \frac{y^2}{36} = 1$

Now the path of the bon is along the ellipse in clockwise direction as shown in **fig. 30(b).**

Ex. 2. *Two vibrations of right angles to one another are described by the equations*

$$x = 10 \cos (5\pi t) \quad \textit{and} \quad y = 10 \cos (5\pi t + \tfrac{\pi}{3})$$

Construct the Lissajous figures of the resulting motion.

Sol. $x = 10 \cos (5\pi t)$ and $y = 10 \cos (5\pi t + \frac{\pi}{3})$

Now, $y = 10 \cos 5\pi t \times \cos \frac{\pi}{3} - 10 \sin (5\pi t) \times \sin \frac{\pi}{3}$

or $y = \frac{x}{2} - \frac{\sqrt{3}}{2}\sqrt{10^2 - x^2}$ $\quad [\because \sin 5\pi t = \sqrt{1-(\cos 5\pi t)^2}\,]$

or $(2y - x)^2 = 3(100 - x^2)$ or $x^2 + 4y^2 - 4xy = -3x^2 + 300$

or $x^2 + y^2 - xy = 75$

whence, $\frac{x^2}{(5\sqrt{3})^2} + \frac{y^2}{(5\sqrt{3})^2} - \frac{xy}{(5\sqrt{3})(5\sqrt{3})} = \mathbf{1.}$

Thus the resulting motion is an oblique ellipse.

Ex. 3. *Two tuning forks produce a Lissajou's figure. The figure changes in form from a parabola to a figure of eight and again to a parabola, the whole cycle occupying* 6 *seconds. If the frequency of one fork is* 100, *find possible frequencies of the other.*

Sol. The figure changes from a parabola to a figure of '8' and then again to a parabola. Evidently then the component frequencies are nearly in the ratio 1 : 2 since the frequency of one tuning fork is 100, that of the other may be nearly 200 or 50.

The same figure being repeated in 6 sec., suggests that in 6 seconds the other fork completes one vibration more or less than double or half the vibrations performed by the first fork. Therefore, in one second the other fork will complete $\frac{1}{6}$ vibration more or less than double or half the vibrations made by the first fork. Obviously, therefore, the other fork will in one second, complete $200 \pm \frac{1}{6}$ or $50 \pm \frac{1}{6}$ vibrations.

Hence the possible frequencies of the other fork are $\mathbf{200 \pm \frac{1}{6}}$ and $\mathbf{50 \pm \frac{1}{6}}$.

Ex. 4. *Two tuning forks whose frequencies are approximately in the ratio* 2 : 1 *are employed to produce Lissajous figures, and it is observed that the figure goes through a cycle of change in* 15 *sec. On loading slightly the fork of the upper pitch, the figure goes through a cycle of changes in* 10 *sec. If the fork of lower pitch has a frequency of* 300 *Hz. calculate the frequency of the other fork before and after loading.*

Sol. Since the frequency of the fork of lower pitch is 200 Hz, that of second may approximately be 600 Hz. Before loading the second fork, the figure goes through a cycle of changes in 15 sec., *i.e.*, the frequency of the second fork may be equal to $600 \pm \frac{1}{15}$.

On loading the second fork, the time of the cycle of changes is reduced to 10 sec. Hence the frequency of the second fork after loading may be $600 \pm \frac{1}{10}$.

But on loading the frequency of the fork must decrease. Therefore, the frequency of the second fork before loading = $600 - \frac{1}{10} = \mathbf{599\frac{14}{15}}$ **Hz.** and after loading = $600 - \frac{1}{10} = \mathbf{599\frac{9}{10}}$ **Hz.**

Ex. 5. *Show that a linear simple harmonic motion can be obtained by the superposition of two equal and opposite circular motions of the same angular frequency.*

Sol. Let us consider two equal and opposite circular motions of the same angular frequency ω [**fig. 31**]. At any time *t*, the position vectors of the rotating point in the two cases are given by

$$\vec{OP_1} = a\cos\omega t\,\hat{\mathbf{i}} + a\sin\omega t\,\hat{\mathbf{j}}$$

$$\vec{OP_2} = a\cos\omega t\,\hat{\mathbf{i}} - a\sin\omega t\,\hat{\mathbf{j}}$$

The superposition of two vectors gives

$$\vec{OP_1} + \vec{OP_2} = 2a\cos\omega t\,\hat{\mathbf{i}}$$

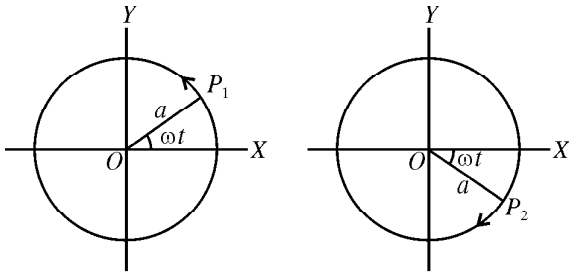


Fig. 31.

Thus the resultant displacement due to the two equal and opposite circular motions is along *X*-axis *i.e.*, straight line and describes a S.H.M. of angular frequency ω.

Ex. 6. *In a cathode ray oscilloscope, the electrons are deflected by two mutually perpendicular electric fields according to the displacement-time relations, given by*

$$x = 4\sin(2\pi\upsilon t + \pi/6) = 4\sin(2\pi\upsilon t)$$

Determine the resultant path followed by the electrons.

Sol.
$$\frac{x^2}{4^2} + \frac{y^2}{4^2} - \frac{2xy}{4\times4}\cos\frac{\pi}{6} = \sin^2\frac{\pi}{6}$$ [See eq. (4), Art. 7.10]

or
$$\frac{x^2}{16} + \frac{y^2}{16} - \frac{2xy}{16}\cdot\frac{\sqrt{3}}{2} = \frac{1}{4} \quad \text{or} \quad x^2 + y^2 - \sqrt{3}xy - 4 = 0$$

Thus the resultant path is an ellipse.

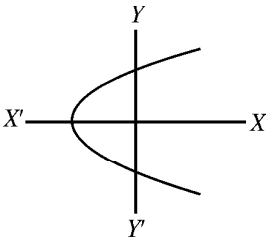


Fig. 32.

Ex. 7. *Lissajous figure in the case of two vibrating forks is a parabola, prove that the frequencies are in the ratio* 1 : 2.

Sol. Suppose that one fork of frequency ν_1, is vibrating along *X*-axis and other of frequency υ_2 along *Y*-axis. Then if the parabola cuts the *X*-axis *m* times and *Y*-axis *n* times **(fig. 32)**, then

$$\frac{\upsilon_2}{\upsilon_1} = \frac{m}{n} = \frac{1}{2} = \mathbf{1 : 2.}$$

14. Superposition of Two Simple Harmonic Motions of Different Frequencies along the Same Line — Modulation and Beats

In physics, there are a number of phenomena in which the motion is a superposition of two simple harmonic motions, having different angular frequencies ω_1 and ω_2. For example, when two tuning forks of different frequencies vibrate simultaneously, they emit sinusoidal waves. The oscillatory motion of the eardrum is, in fact, a superposition of two simple harmonic motions due to the two waves. To simplify mathematically, it is assumed that the two harmonic motions have the same amplitude *a* so that the phase difference $[(\omega_2 t + \phi_2) - (\omega_1 t + \phi_1) = (\omega_2 - \omega_1)t + \phi_2 - \phi_1]$ between the two vibrations is changing continuously with time. Since the initial phase difference $(\phi_2 - \phi_1)$ does not play any significant role in this case, we assume that the two oscillations have zero initial phase and are represented as

$$x_1 = a\sin\omega_1 t \text{ and } x_2 = a\sin\omega_2 t \qquad ...(1)$$

The resultant motion is obtained as a superposition of the two motions i.e.,

$$x = x_1 + x_2 = a\sin\omega_1 t + a\sin\omega_2 t \qquad ...(2)$$

Modulation

Eq. (2) can be written as

$$x = 2a\cos\left(\frac{\omega_1-\omega_2}{2}\right)t\ \sin\left(\frac{\omega_1+\omega_2}{2}\right)t \qquad ...(3)$$

which represents an oscillatory motion with angular frequency $\frac{\omega_1+\omega_2}{2}$ and amplitude $2a\ \cos\left(\frac{\omega_1-\omega_2}{2}\right)t$.

We define an **average angular frequency** ω_{av} and a **modulated angular frequency** ω_{mod} by

$$\omega_{av} = \frac{\omega_1+\omega_2}{2} \text{ and } \omega_{mod} = \frac{\omega_1-\omega_2}{2} \qquad ...(4)$$

and a **modulation amplitude** $a_{mod}(t)$ by

$$a_{mod}(t) = 2a\cos\left(\frac{\omega_1-\omega_2}{2}\right)t = 2a\ \cos\ \omega_{mod}\ t \qquad ...(5)$$

Therefore the superposed oscillation is given by

$$x = a_{mod}(t)\ \sin\ \omega_{av}(t) \qquad ...(6)$$

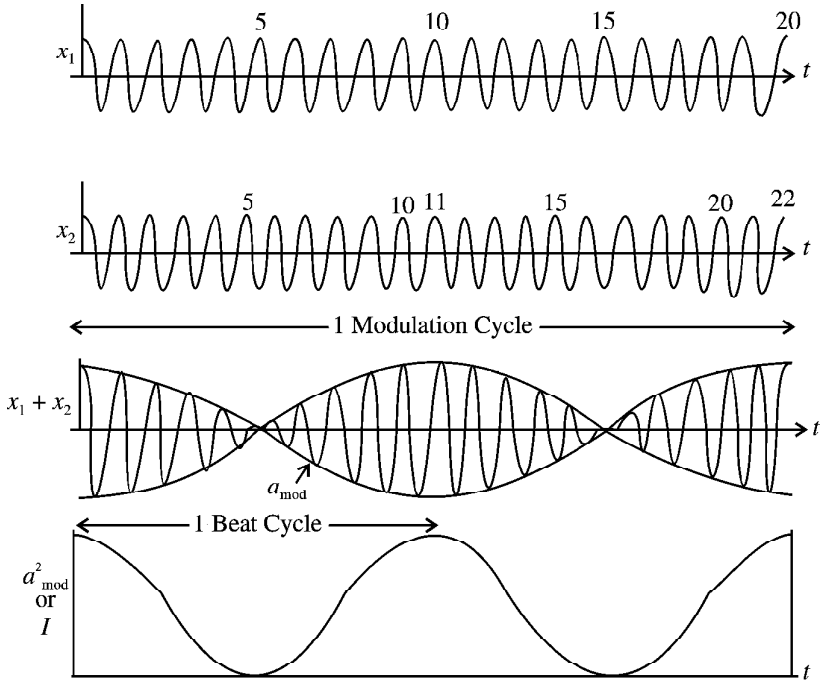


Fig. 33: Modulation an formation of beats.

The superposition of two simple harmonic motions of different frequencies is shown in **fig. 8.8**. The figure also shows the variation of the modulation amplitude with time.

We see from eq. (5) that the modulation amplitude varies with time with a frequency, given by

$$\upsilon_{mod} = \frac{\omega_{mod}}{2\pi} = \frac{(\omega_1-\omega_2)/2}{2\pi} = \frac{\upsilon_1-\upsilon_2}{2} \qquad ...(7)$$

In a cycle, the modulation amplitude in eq. (6) is $2a$, 0, $-2a$ and 0, as the phase angle ($\omega_{mod}\ t$)

becomes 0, $\pi/2$, π and $3\pi/2$ respectively. However the intensity (I) is proportional to the square of the amplitude, it will be maximum two times and also minimum two times in a cycle of the modulation amplitude. For example, in case of two sound waves (of equal amplitudes), the ear will hear the maximum or minimum sound at double the frequency of the modulated amplitude. *This fluctuation in the intensity of sound is called* **beats**. The frequency of the beats is given by

$$\upsilon_{beat} = 2\upsilon_{mod} = \upsilon_1 - \upsilon_2 \quad ...(8)$$

Mathematically, we obtain this result as follows:

$$I \propto a^2_{mod}(t) = 4a^2\cos^2 \omega_{mod}(t) = 2a^2 (1 + \cos 2\omega_{mod} t) \quad ...(9)$$

Therefore $a^2_{mod}(t)$ or I oscillates at angular frequency $2\omega_{mod}$ *i.e.,* the frequency of the beats is twice the modulation frequency. However one can hear the beats when $|\upsilon_1 - \upsilon_2|$ is less than about 10 Hz.

Ex. 1. *Two vibrations along the same line are represented by the equations*

$$x_1 = a \sin 20\ \pi t \text{ and } x_2 = a \sin 22\pi t.$$

Deteremine the beat frequency and draw a graph of the resultant motion over one beat period.

Sol. Here, ν_1 = 10 Hz and ν_2 = 11 Hz. Beat frequency = $|\nu_1 - \nu_2|$ = **1 Hz.**

$\therefore$ For graph see **fig. 7.33.**

QUESTIONS

Long Answer Questions

1. What is simple harmonic oscillator? Deduce the equation of motion of simple harmonic oscillator and obtain its solution. **(A.N.U. 2009; Krishna U. 2014; Dr. B.U. 2013)**

2. What is simple harmonic motion? Mention its characteristics. Represent displacement, velocity and accleration in a single diagram. **(A.U. 2012; A.N.U. 2007)**

3. What is the simple harmonic motion? Write its characteristics. Establish the equation of motion of a simple harmonic oscillator and solve it. **(S.K.U. 2003)**

4. Define simple harmonic motion and explain its characterstics. Derive expressions for the frequency and energy of a simple harmonic oscillator. **(A.N.U. 2012; O.U. 2013; A.U. March 2007)**

5. (a) Define simple harmonic motion and explain its physical characteristics. **(O.U. March 2007, K.U. April 2005)**

(b) Show that the frequency of the simple harmonic oscillator is independent of the amplitude and its energy is proportional to the square of the amplitude.

6. (a) What are the characteristics of a simple harmonic oscillator?

(b) Obtain an expression for the total energy of a simple harmonic oscillator. Show that the total energy is independent of the instantaneous displacement of the body. **(O.U. 2002)**

7. Show that the total energy of a simple harmonic oscillator is directly proportional to the square of its amplitude. **(O.U. 2002)**

8. Derive an expression for the energy of a body executing simple harmonic motion and show that it is directly proportional to the square of the amplitude and inversely proportional to the square of the period.

9. Describe the equation of motion of a simple harmonic oscillator. Obtain its solution. What are its characteristic? Show that the total energy of simple harmonic oscillator is proportional to the square of the amplitude. **(A.U. 1995)**

10. Show that solution of the equation $\frac{d^2x}{dx^2} + \omega^2 x = 0$ is of the form $x = a \sin(\omega t + \phi)$.

Hence prove that for initial displacement x_0 and initial velocity v_0, the amplitude $a = \sqrt{x_0^2 + v_0^2/\omega^2}$ and initial phase $\phi = \tan^{-1}(\omega x_0/v_0)$.

11. Set up the differential equation for the vibration of a spring treating it as a simple harmonic oscillator and obtain its solution. Discuss how the frequency of vibration of the spring is modified, if the mass of spring is taken into consideration. Does the force of gravity has any effect on the frequency of vibration of the spring? **(A.N.U. 1990)**

12. Derive an expression for the frequency of vibration of a loaded spring taking its mass into consideration. **(S.K.U. 2007)**

13. What is simple harmonic motion? Obtain the expression for the frequency of vibration of a loaded spring taking its mass into consideration. **(A.N.U. March 2007; S.V.U. 2007)**

14. Define compound pendulum. Derive the expression for acceleration due to gravity using compound pendulum. **(A.N.U. 2013)**

15. What is a compound pendulum? Explain with necessary theory, how the value of acceleration due to gravity 'g' can determined using compound pendulum. **(A.N.U. 2009; S.V.U. 2011; S.K.V. 2011)**

16. What is a pendulum? Define torsion pendulum and how you determing modulus of rigidity by using torsion pendulum. Expalin. **(S.V.U. 2009)**

17. Define simple harmonic motion. Explain the torsional pendulum and obtain an expression for rigidity modulus of the material of a given wire. **(O.U. 2010; A.N.U. 2010)**

18. Discuss the linear combination of two mutually perpendicular simple harmonic vibrations of equal frequencies. **(S.K.U. 2014; S.V.U. 2014)**

19. Derive an expression for amplitude and phase difference for a linear combination of two simple harmonic motions in same direction having the same frequency. **(O. U. 2013)**

20. Discuss the linear combination of two S.H.M.'s, which are (i) with same angular frequency (ii) with slightly different frequencies. Explain how, the resultant phase and amplitudes change? **(O.U. 2003)**

21. Discuss with mathematical theory the linear combination of simple harmonic vibrations of the same frequency. **(A.U. 2003)**

22. Discuss the effect of combining two S.H.M. in the same direction of the same angular frequency but different amplitudes and phases. How does the amplitude and phase of the resultant vibration get affected if two frequencies are slightly different from each other. **(O.U. 1990)**

23. Find the resultant vibration when two S.H.M. at right angles and differing in phase by 45º, but same frequency are compounded. **(K.U. 1991)**

24. Find the resultant of two S.H.M.'s. of periods in the ratio 1 : 2 and having a phase difference of 90º. **(O.U. 2010)**

25. Define simple harmonic motion. Write any four characteristics of simple harmonic motion. Discuss the theory of linear combination of two simple harmonic motions of the same frequency. **(S.K.U. 2002)**

26. Define simple harmonic motion. Give examples. Discuss the linear combination of two mutually perpendicular simple harmonic vibrations of equal frequencies. **(S.V.U. 2010)**

27. Find the resultant of two simple harmonic motions of equal periods when they act at right angles of each other and discuss few cases. **(S.K.U. 2007)**

28. Explain Lissajous figures when two simple harmonic oscillations of the frequency ratio 1 : 1 and phase difference varying between 0 and π, superpose.

29. When two mutually perpendicular simple harmonic motions of amplitudes and time periods in the ratio 1 : 1, but phase difference (i) 90º, (ii) 45º superpose, deduce the expression for the figures obtained.

30. What are Lissajous figures? Find the resultant of two simple harmonic motions of the same frequency acting along the same line but different phase. **(O.U. 2009; S.V.U. 2003)**

31. Discuss analytically the linear combination of two simple harmonic motions of the same frequency acting at right angles with various phase differences. **(A.U. June 2014, 07; A.N.U. 2011)**

32. If two mutually perpendicular simple harmonic oscillations represented by equations $x = a \sin(2\omega t + \phi)$ and $y = b \sin \omega t$., superpose simultaneously on a particle, deduce the equation for the resultant path of the particle. What will be the shape of the path if $\phi = 0^o$, $\pi/2$ and π?

33. Two perpendicular simple harmonic motions, whose amplitudes are different but the frequencies are in the ratio 2 : 1, are superposed together. Discuss those conditions in which the phase difference is 0, $\pi/2$ and π.

34. What are Lissajou's figures? Find the resultant of simple harmonic motions of equal period's when they act at right angles to one another. Discuss the different important cases. **(A.N.U. 2008; S.V.U. 2005)**

35. What are Lissajous's figures? Find the resultant of two mutual perpendicular simple harmonic vibrations of same frequency. **(Dr. B.R.A.U. 2014; O.U. Oct. 2007)**

36. What are Lissaou's figures? How they can be obtained experimentally? Give some important uses of Lissaous figures.

Short Answer Questions

1. Define S.H.M. and give three examples.? **(S.V.U. 2009; A.N.U. 2010)**

2. What is S.H.M.? What are its characteristics? **(S.V.U. 2014; A.N.U. 2009; S.K.U. 2005)**

3. Describe the characteristics of linear harmonic oscillator. **(A.N.U. 2003)**

4. Discuss the physical significance of simple harmonic motion. **(A.N.U. 2001)**

5. Explain how the change of velocity of a S.H. oscillator changes with position and time? **(S.V.U. 2002)**

6. Derive an expression for the energy of the harmonic oscillator. **(A.N.U. 2008; A.U. 2001)**

7. Describe how the energy of an S.H.O. changes during an oscillation. **(S.V.U. 2002)**

8. At what displacement the potential and kinetic energies of a simple harmonic motion are equal. **(O.U. 2001)**

9. Express the kinetic and potential energies as a function of total energy of harmonic oscillator when its displacement half the amplitude.

[**Ans.** $K = \frac{3}{4}E, U = \frac{1}{4}E.$]

10. Show that the total energy of a simple harmonic oscillator is directly proportional to the square of its amplitude. **(K.U. 1995)**

11. Differentiate between the periodic motion and simple harmonic motion, giving two examples of each.

[**Ans.** If a moving particle retraces its path regularly after an interval of time, its motion is said to be periodic. Examples: (i) uniform circular motion of a particle, (ii) motion of a planet on elliptical path, (iii) motion of a simple pendulum etc. Simple harmonic motion is a particular type of periodic motion, in which at any instant the restoring force is proportional to displacement and directed opposite to it. Examples: motion of simple pendulum, spring-mass system, bar pendulum etc.]

12. A particle of mass 0.1 kg lies in a potential field $V = 50x^2 + 100$ J/kg. What is the force on the particle ? Write the differential equation of motion.

[**Ans.** Force $F = -10x$ newton; $\frac{d^2x}{dt^2} + 100x = 0.$]

13. Find the frequency of oscillation in Q. 12.

[**Ans.** $5/\pi.$]

14. Show that in a compound pendulum, the centre of suspension and centre of oscillation are interchangeable.

15. Show that in simple harmonic motion the displacement of the particle is the same after a time $2\pi/\omega$.

16. The equation of a particle executing simple harmonic motion is

$x = (1\text{ m})\sin\left[(\pi s^{-1})t + \frac{\pi}{3}\right]$.

Write down the amplitude, time period and maximum speed.

[**Ans.** $a = 1$ m, $T = 2s$, $v_m = 5\pi$ m/s.]

17. A particle executing S.H.M. comes to rest at the extreme positions. Is the resultant force on the particle zero at these postions? Explain.

[**Ans.** No.]

18. Two particles execute simple harmonic motions of the same amplitude and frequency on parallel lines side by side. They are cross one another when moving in opposite directions each time their displacment is half their amplitude. What is the phase difference between them?

[**Ans.** Phase difference = $2\pi/3$.]

19. In S.H.M. when the displacement is one-half the amplitude, what fractions of the total energy are kinetic and potential?

[**Ans.** K/E = 0·75 and U/E = 0·25.]

20. At what displacement the energy half kinetic and half potential?

[**Ans.** $a/\sqrt{2}$.]

21. Given $x = 2.5\sin\left(\frac{2\pi t}{3} + \frac{\pi}{2}\right)$, then find the values of angular velocity, maximum linear velocity, initial phase and time period.

[**Ans.** 2.09 radian/sec, 5.22 units, $\pi/2$, 3 sec.]

22. Derive an equation for the frequency of vibration of a spring taking its mass into consideration. **(A.U. 2005; A.N.U. 2005; S.K.U. 2003; S.V.U. 2002)**

23. Discuss about the amplitude for linear combination of two simple harmonic oscillators of same frequency. **(A.V.U. 2005)**

24. What are Lissajous' figures. **(S.K.U. 2014; K.U. 2012; S.V.U. 2007)**

25. Write a short note on Lissajous's figures. **(S.V.U. 2011; A.N.U. 2009, A.U. 2005)**

26. What are 'Lissajous figures'? Mention their applications. **(K.U. 2014; O.U. 2014)**

27. Write some important uses of Lissajous figures. **(Y.V.U. 2014)**

28. What are Lissajous figures? Give an example. **(S.K.U. 1996; S.V.U. 1996)**

29. Write a note on the combination of two S.H.M. along a line. **(S.K.U. 1996)**

30. What are beats? How they are produced? **(A.N.U. 2001; S.V.U. 2003)**

31. If the Lissajous figure has the shape of letter '8', what is the ratio of component frequencies?

[**Ans.** 2 : 1.]

32. What is the shape of Lissajous figure, when two mutually perpendicular simple harmonic motions of same amplitude and frequency are superposing on each other with a phase difference of $\pi/2$.

[**Ans.** Circle.]

33. Find the resultant of two simple harmonic motions of same frequency acting along two mutually perpendicular directions. **(S.U. 2014)**

PROBLEMS

[Simple Harmonic Motion]

1. A S.H.M. is represented by the equation
$x = 0.12 \cos (12t - \pi/6)$
where x is in metre and t in second. Find the frequency, time period and maximum velocity.
(A.U. 2002)
[**Ans.** $\nu = 1.91$ Hz; 0.52 sec; $v_{max} = 1.44$ m/sec.]

2. A particle executes simple harmonic motion with a period of 0.002 sec and amplitude 10 cm. Find its acceleration when it is at a distance of 4 cm from its mean position.
(R.U. 2014; A.U. 2001, S.V.U. 2003)
[**Ans.** -3.94×10^5 m/sec^2]

3. The displacement equation of a particle describing simple harmonic motion is $x = 0.01 \sin 100\ \pi (t + 0.005)$ metre, where x is the displacement of the particle at any instant t. Calculate the amplitude, periodic time, maximum velocity and displacement at the time of start of the motion.
[**Ans.** $a = 0.01$ m, $T = 0.02$ sec, $v_{max} = 3.14$ m/sec, $x_0 = 0.01$ m]

4. A particle executes S.H.M. with a period of 0.002 sec and amplitude 10 cm. Find its accleration when it is 4 cm away from its mean position and also obtain its maximum velocity.
[**Ans.** -3.9×10^5 m/sec^2; 314 m/sec.]

5. The displacement equation for a particle in simple harmonic motion at an instant t is,
$x = 0.01 \sin 100\ \pi(t + 0.005)$ m
Calculate (i) amplitude; (ii) time period; (iii) maximum velocity and (iv) displacement at start.
[**Ans.** (i) 0.01 m; (ii) 0.02 sec.; (iii) 3.14 m/s; (iv) 0.01 m.]

6. A particle makes S.H.M. along a straight line and its velocity when passing through points 3 cm and 4 cm from the centre of its path is 16 cm/sec and 12 cm/sec respectively. Find (a) the amplitude; (b) the time period of motion.
[**Ans.** 0.05 m; 1.571 sec.]

7. Prove that the velocity of a particle in S.H.M. at a distance $\sqrt{3}/2$ of its amplitude from the centre is half of its velocity at the central position.

8. A test tube weighs 6 gm and has an external diameter of 23 cm is floated in water by placing 10 gm mercury at the bottom of the tube. The tube is depressed by a small amount and then released. Find its time period.
[**Ans.** 0.453 sec.]

9. The displacement of a particle executing S.H.M. is given by

$$x = 20 \sin\left(\frac{2\pi t}{T} + \phi\right)$$

The period of vibration is 60 seconds. At $t = 0$, the displacement of the particle is 10 cm. Calculate (i) the initial phase, (ii) the phase angle when the displacement is 4 cm and (iii) the phase difference between any two positions of the particle 10 seconds apart.
[**Ans.** (i) $\pi/6$, (ii) $\sin^{-1}(0.15)$, (iii) $\pi/3$]

10. If a smooth tunnel is bored through the earth along one of its diameters, then calculate the time of one oscillation of the ball. ($R = 6.4 \times 10^3$ km and $g = 9.8$ metres/sec^2).
[**Ans.** 84 minutes.]

11. A particle in S.H.M. has velocities u_1 and u_2 when displacements from the mean positions are x_1 and x_2 respectively. Calculate period, amplitude and maximum speed of the particle.

[**Ans.** $T = 2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{u_1^2 - u_2^2}}$, $a = \sqrt{\dfrac{u_1^2 x_2^2 - u_2^2 x_1^2}{u_1^2 - u_2^2}}$, $v = \sqrt{\dfrac{u_1^2 x_2^2 - u_2^2 x_1^2}{x_2^2 - x_1^2}}$.]

12. A particle of mass m executes simple harmonic motion of amplitude a and frequency n. Find the force constant, the maximum kinetic energy and the average kinetic energy.
[**Ans.** $4\pi^2n^2m$, $2m\pi^2n^2a^2$, $m\pi^2n^2a^2$.]

13. If the potential energy of a harmonic oscillator in its resting position is 10 joules and the average kinetic energy is 5 joules, what is its total energy at any instant?
[**Ans.** 20 joules.]

[Spring-Mass System]

14. A body of mass 0.25 kg hangs from a spring and oscillates. Find the time period and frequency. [Spring force constant = 40 Nm^{-1}]. **(A.U. 2010, 09; S.V.U. 2003; A.N.U. 2001)**
[**Ans.** 2.014 sec.]

15. (a) A vertical spring extends by 25 cm, when mass of 5 kg loads it. Calculate the frequency of oscillation, if the spring is loaded with 4 kg and allowed to oscillate. **(S.V.U. 2012, 03; O.U. 2003)**
[**Ans.** 1.115 Hz]

(b) A spring of force constant 20 N/m is hung vertically and loaded with a mass 0.1 kg and allowed to oscillate. Calculate the time period and frequency of oscillation of the spring. **(A.N.U. 2009)**
[Ans. T = 0.44 sec.; v = 2.26 Hz]

16. A body of mass 0.2 kg is hung from a spring of constant 100 N/m. Calculate the value of undamped frequency. **(Krishna U. 2014; A.N.U. 05)**
[**Ans.** 3.56 Hz]

17. An object of mass 2.5 kg is hanging from a massless spring. Another object of mass 400 gm, hung below the first object, stretches the spring 2 cm farther. If the object of 400 gm is removed and the first object is set into vertical oscillations, determine the period of oscillation.
[**Ans.** 0.71 sec.]

18. A body of mass 4.9 kg when suspended from a spring, oscillates with a period 0.5 sec. By what amount will the spring be contracted on the removal of the body?
[**Ans.** 0.062 m.]

19. One end of a vertical spring is attached to a fixed support and to the lower end a mass m is suspended. If by suspending the mass, the lower end of the spring is stretched through x_0 distance, then find the spring constant and period of oscillation.

[**Ans.** mg/x_0; $2\pi\sqrt{x_0/g}$.]

[**Hint:** $mg = kx_0$.]

20. (a) Two springs are joined and connected to mass M as shown in **fig. 12**. If the springs separately have force constants k_1 and k_2, show that the frequency of oscillation is $n = \frac{1}{2\pi}\sqrt{\frac{k_1k_2}{(k_1+k_2)m}}$.

(b) If the two massless springs are attached to a mass M and to fixed supports as shown in **fig. 11**, then show that the frequency of oscillation is given by $n = \frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{m}}$.

21. In the in **fig. 11** M is a mass which is placed between two rigid supports A and B, and is controlled by two massless springs. If the mass of M is 50 gms and the force constants of springs are 3000 and 2000 dynes/cm, deduce:
(i) the 'frequency' of small oscillations of the mass M.
(ii) the energy of vibration of amplitude 0.4 cm.
(iii) the velocity of mass M when it passes through its mean position.
[**Ans.** (i) 1.59 Hz; (ii) 4×10^{-5} joules; (iii) 0.04 m/sec.]

22. Find the period and frequency for arrangement shown in **fig. 33**:

Fig. 33.

The two springs have force constants k_1 and k_2.

[**Ans.** (i) $T = 2\pi\sqrt{\dfrac{(k_1+k_2)m}{k_1k_2}}$ $\nu = \dfrac{1}{2\pi}\sqrt{\dfrac{k_1k_2}{(k_1+k_2)m}}$ (ii) $T = 2\pi\sqrt{\dfrac{m}{k_1k_2}}; \nu = \dfrac{1}{2\pi}\sqrt{\dfrac{k_1+k_2}{m}}$

[Compound Pendulum]

23. A sphere of radius R is suspended by a string from a fixed point, so that the distance from the centre of sphere to the point of suspension is ($l >> R$). Show that the period of the pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{R^2}{5l^2}\right).$$

If R = 0.05 m and l = 1 m, what is the length of the equivalent simple pendulum?
[**Ans.** 1.001 m.]

24. A body of mass 200 gm oscillates about a horizontal axis at a distance of 20 cm from its centre of gravity. If the length of the equivalent simple pendulum be 35 cm, find its moment of inertia about the axis of suspension.
[**Ans.** 1.4×10^{-4} kg $\times$ m^2.]

25. A thin uniform bar of length 1.2 m is made to oscillate about an axis through its end. Find the period of oscillation and other points about which it can oscillate with the same period.
[**Ans.** 1.796 sec; Points are situated at distance 0.4, 0.8, 1.2 m from one end.]

26. (i) Assuming small amplitude oscillations, what would be the frequency of a simple pendulum of length 2 metres, in an elevator accelerating upward at a rate of 2.0 metres/sec^2?

(ii) What would be its frequency in free fall?

[**Ans.** (i) 0.38 Hz; (ii) Zero.]

27. Prove that a uniform rod of length L hanging as a compound pendulum from a pivot at one end has the same frequency as a simple pendulum of length $2L/3$. for oscillations of small amplitude.

28. A thin rod of length 60 cm is used as a compound pendulum. Find the position of a point such that the period about it is a minimum.

[**Ans.** 17.3 cm from C.G.]

29. A disc of radius r oscillates as a pendulum about a horizontal axis perpendicular to the plane of the disc through a point $r/2$ from its centre. Calculate the period of oscillation.

[**Ans.** $T = 2\pi\sqrt{3r/2g}$.]

30. A metal disc of radius R is made to oscillate about an axis passing through a point on its edge. Find (a) the equivalent length of simple pendulum, and (b) the minimum time period of oscillation.

[**Ans.** $3R/2$; $T = 2\pi\sqrt{3R/2g}$.]

31. Calculate the time of oscillation of a uniform square lamina of side 30 cm about an axis perpendicular to its plane and passing through one of its corners. (g = 9.8 m/sec^2).

[**Ans.** $T = \pi\sqrt{13/98}$.]

32. A uniform circular rod with a radius of 2 cm oscillates when suspended from a point on its axis at a distance of 4 cm from one end. If the length of the rod is 1 metre, find the point or points from which, it suspended, the periodic time would remain unaltered.

[**Ans.** At 31.87 cm, 68.13 cm and 96 cm from the same end.]

33. A uniform rod *AB* of mass 100 gm and length 120 cm can swing in a vertical plane about *A* as pendulum. A particle of mass 200 gm is attached to the rod at a distance *x* from *A*. Find *x* such that the period of vibration is a minimum.

[**Ans.** $x = 2.748$ cm.]

[Torsional Pendulum]

34. A solid cylinder of 2 cm radius weighing 200 gm is rigidly connected, with its axis vertical, to the lower end of a fine wire. The period of oscillation of the cylinder under the influence of the torsion of wire, is 2 seconds. Calculate the couple necessary to twist it through four complete turns.

[**Ans.** 9.9×10^{-3} N-m.]

35. A torsion pendulum consists of a rectangular wood block 8 cm × 12 cm × 3 cm and with a mass 0.3 kg suspended by means of a wire passing through its centre in such a way that the shortest side is vertical. If the period of torsional oscillations is 2.4 sec, find the torsion constant of the wire.

[**Ans.** 3.565×10^{-3} N-m/rad.]

36. A solid spherical bob of radius 0.05 m and mass 3.5 kg is suspended from a wire of torsional rigidity 6×10^{-3} N-m/rad to form a torsion pendulum. Find the period. Find also the period, in case if a hollow sphere of the same mass and same material but with external radius 6 cm were used.

[**Ans.** 4.8 sec; 6.62 sec.]

37. The balance wheel of a watch vibrates with an angular amplitude of π radians and a period of 0.5 second. Determine (i) the maximum angular speed of the wheel, (ii) the angular speed of the wheel at the displacement $\pi/2$ radians, and (iii) the angular acceleration of the wheel at the displacement $\pi/4$ radians.

[**Ans.** (i) 39 rad/sec., (ii) 34 rad/sec., (iii) 120 rad/sec^2.]

[Combination of Two Perpendicular S.H.M.'s]

38. Two forks are used to produce Lissajous figures. It is found that the figure completes its cycle in 10 sec. If the frequency of *A* (know to be slightly longer than that of (*B*) is 200. Calculate the frequency of *B*.

[**Ans.** 199.9 Hz.]

39. Two tuning forks are vibrated perpendicularly and a beam of light is reflected from the mirrors, attached to the prongs of the forks. The elliptical figures are produced on the screen and they change in shape as time progresses. The figures complete a cycle of changes in 10 seconds. If the frequency of one fork is 250 (slightly less than that of the other), determine the frequency of the other.

[**Ans.** 250.1 Hz.]

40. Two tuning forks *P* and *Q* of frequencies close to each other are used to obtain Lissajous figures and it is observed that the figures goes through a cycle of changes in 20 seconds. Now if *P* is loaded slightly with wax, the figure goes through a cycle of changes in 10 seconds. If the frequency of *Q* is 300 Hz, what is the frequency of *P* before and after loading.

[**Ans.** 299.95 Hz; 299.9 Hz.]

41. Two rectangular simple harmonic vibrations of frequency ratio 2 : 1 are represented as follows:

$$x = a_1 \cos (2\omega t + \phi) \text{ and } y = a_2 \cos \omega t$$

Find the resultant motion, when (i) $\phi = 0$; (ii) $\pi/2$ and (iii) π.

[**Ans.** (i) Parabola; (ii) Figure of '8'; (iii) Parabola oppositely directed to that of (i).]

●

2

Damped and Forced Oscillations

1. Introduction

A particle will perform simple harmonic motion with a costant amplitude (or energy) of vibration, provided the restoring force on it is proportional to the displacement ($F = -kx$). In actual practice, we observe that if a physical system, once oscillated, its oscillations die out with the passage of time. For example, if we oscillate a pendulum, the amplitude of vibration does not remain constant but decreases gradually with time and ultimately the pendulum comes to rest. In fact, when a body moves in a medium, frictional foces act on the body in a direction opposite to its movement and the work done against the frictional forces results in the decrease of its mechanical energy. This is why, an oscillating pendulum loses energy and consequently its amplitude of oscillation becomes smaller and smaller as the time elapses.

In the presence of frictional forces, the motion of the oscillator is said to be **damped** by friction and the vibrating system is called **damped harmonic oscillator**. The frictional or damping force on the oscillator arises from the retarding action of the medium. The magnitude of this force is some function of the velocity of the moving body. At ordinary speeds, the damping force (F_d) is found proportional to the velocity (v) in a direction opposite to it i.e., $F_d = -\gamma v$, where γ is a constant called the damping coefficient. It depends on the size of the body.

2. Damped Harmonic Oscillator

In the presence of frictional forces, arising due to the viscosity of the medium, the oscillatory system is called **damped harmonic oscillator**. The free oscillations of a physical system in the presence of frictional forces are callled **damped harmonic motions**. When damping is taken into account, then a harmonic oscillator is acted by the following forces:

(1) A restoring force $-kx$ at the displacement x due to the spring action, (K = force constant) and

(2) A damping force $-\gamma v$, where $v = dx/dt$ is the velocity of the oscillator at the displacement x and the constant γ is the *damping coefficient or frictional force per unit velocity.* This γ is also called **mechanical resistance.**

Thus the resultant force on the oscillating body is given by

$$F = -kx - \gamma \frac{dx}{dt}$$

Using Newton's second law, the equation of motion of the damped harmonic oscillator is given by

$$m\frac{d^2x}{dt^2} = -kx - \gamma \frac{dx}{dt} \qquad ...(1)$$

where m is the mass of the oscillating body.

Dividing by m, eq. (1) can be written as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \qquad ...(2)$$

where $2b = \gamma/m$ and $\omega_0^2 = k/m$. Here $\omega_0 = \sqrt{k/m}$ is the natural angular frequency of the oscillator in absence of the damping force. The constant $b = \gamma/2m$ is called **damping or decay constant** and the

constant $\frac{1}{b} = \frac{2m}{\gamma}$ is termed as **decay modulus**. Also if we put $m/\gamma = \tau = 1/2b$, then τ is called **relaxation time** or **time constant**.

Equation (2) is called the **differential equation of damped harmonic oscillator**.

Examples of Damped Harmonic Oscillator

(1) A spring-mass system, when displaced from equilibrium, oscillates with decreasing amplitude with time and finally comes to rest.

(2) A pendulum, disturbed from equilibrium position, oscillates with decreasing amplitude with time and ultimately acquires equlibrium position.

(3) Charge oscillations in a *LCR* circuit die away after some time.

(4) Oscillations of a ballistic galvanometer coil with decreasing amptitude.

Solution of the Differential Equation of Damped Harmonic Oscillator

Let the solution of eq. (2) be

$$x = Ae^{\alpha t}$$

Substituting the values of x, dx/dt and d^2x/dt^2 in eq. (2), we have

$$(\alpha^2 + 2b\alpha + \omega_0^2)e^{\alpha t} = 0 \text{ or } \alpha^2 + 2b\alpha + \omega_0^2 = 0$$

$$\therefore \quad \alpha = -b \pm \sqrt{b^2 - \omega_0^2}$$

Hence the solutions of the eq. (2) are

$$x = A_1 e^{(-b+\sqrt{b^2-\omega_0^2})t} \text{ and } x = A_2 e^{(-b-\sqrt{b^2-\omega_0^2})t}$$

Thus the most general solution of eq. (2) is

$$x = A_1 e^{(-b+\sqrt{b^2-\omega_0^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega_0^2})t}$$

i.e.,
$$x = e^{-bt}(A_1 e^{(\sqrt{b^2-\omega_0^2})t} + A_2 e^{(-\sqrt{b^2-\omega_0^2})t}) \quad ...(4)$$

where A_1 and A_2 are constants, depending upon the initial conditions of motion.

The quantity $\sqrt{b^2 - \omega_0^2}$ is imaginary, real or zero, it depends on the relative values of b and ω_0. If $b < \omega_0$, then $\sqrt{b^2 - \omega_0^2}$ is imaginary and it is called **underdamped case**. If $b > \omega_0$ or $b = \omega_0$, the situations are known as **overdamped** and **critically damped** respectively.

(i) Underdamped case/($b < \omega_0$) i.e.**, case of damped harmonic oscillations:** If the damping is so low that $b < \omega_0$, then $\sqrt{b^2 - \omega_0^2} = \sqrt{-1(\omega_0^2 - b^2)} = i\sqrt{\omega_0^2 - b^2} = i\,\omega$, where $i = \sqrt{-1}$ and $\omega = \sqrt{\omega_0^2 - b^2}$, which is a real quantity. Now, from eq. (4), we have

$$x = e^{-bt}[A_1 e^{i\omega t} + A_2 e^{-i\omega t}] \quad = e^{-bt}[A_1(\cos \omega t + i \sin \omega t) + A_2 (\cos \omega t - i \sin \omega t)]$$

or
$$x = e^{-bt}[(A_1 + A_2) \cos \omega t + i (A_1 - A_2) \sin \omega t]$$

As x is real quantity, $(A_1 + A_2)$ and $i(A_1 - A_2)$ must be real quantities. Clearly, A_1 and A_2 are complex quantities. If $A_1 + A_2 = a_0 \sin \phi$ and $i(A_1 - A_2) = a_0 \cos \phi$, then

$$x = a_0 e^{-bt} \sin(\omega t + \phi) \quad ...(5)$$

This equation represents a **damped harmonic motion**. This motion is oscillatory or ballistic whose **periodic time** is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} \quad ...(6)$$

Thus, the effect of damping is to increase the periodic time. If $b = 0$, i.e., no damping, then

$T = 2\pi/\omega_0$. But in actual cases, the effect of damping on periodic time is usually negligible except few extreme cases.

The **amplitude** of the damped harmonic oscillations [eq. (5)] is given by

$$a = a_0 e^{-bt} = a_0 e^{-t/2\tau} \quad ...(7)$$

where a_0 is the amplitude in the absence of damping. In the presence of damping, the amplitude decreases exponentially with time. In **fig 1**, time displacement curve is shown for damped harmonic motion. As the maximum value of sin $(\omega t + \phi)$ is alternately +1 and –1, obviously the time displacement curve of the vibrating body lies entirely between the curves $a = a_0 e^{-bt}$ and $a = -a_0 e^{-bt}$, shown by the dotted lines.

Logarithmic decrement: The time interval between two successive maximum displacements (i.e., amplitudes) of left hand and right hand sides is $T/2$. If a_n and a_{n+1} are the two successive amplitudes, then

$$a_n = a_0 e^{-bt} \text{ and } a_{n+1} = a_0 e^{-b(t + T/2)}$$

Therefore, $$\frac{a_n}{a_{n+1}} = \frac{e^{-bt}}{e^{-b(t+T/2)}} = e^{bT/2} \quad ...(8)$$

The quantity $e^{bT/2}$, say d, is a constant of motion, representing reduction in the amplitude and hence we call it **decrement.**

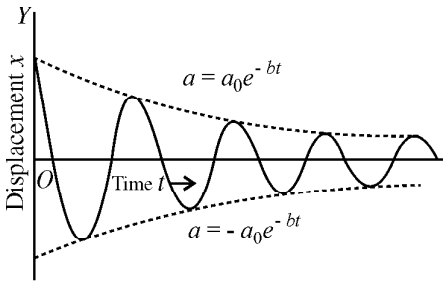


Fig. 1: Variation of amplitude with time.

Now, $$d = e^{bT/2} = \frac{a_n}{a_{n+1}}; \therefore \log_e d = \log_e \frac{a_n}{a_{n+1}} = \frac{bt}{2} = \lambda \text{ (say)} \quad ...(9)$$

We call λ as the **logarithmic decrement** which is equal to the natural logarithmic of the ratio of two successive amplitudes of vibration.

If a_1 is the first maximum displacement, occurring after $T/4$ time, then

$$a_1 = a_0 e^{-bT/4} \text{ or } a_0 = a_1 e^{bT/4} \quad ...(10)$$

From which one can determine the amplitude a_0 in absence of damping.

(ii) Overdamped case: If the damping is high i.e., $b > \omega_0$, then $\sqrt{b^2 - \omega_0^2}$, say β, is a real quantity and then from eq. (4), we have

$$x = (A_1 e^{-(b-\beta)t} + A_2 e^{-(b+\beta)t}) \quad ...(11)$$

As $b > \beta$, both quantities of right hand side decrease exponentially with time and the displacement comes to zero in course of time **[fig. 2]**. In case of high damping the motion is **non-oscillatory**. Such a motion is called **dead beat** or **aperiodic** or **overdamped**. Such a motion is realized practically (i) when a pendulum moves in a very viscous liquid, (ii) in a moving coil galvanometer with a low resistance across its terminals, etc.

(iii) Critically damped case: If $b = \omega_0$, then from eq. (4), we have

$$x = (A_1 + A_2)e^{-bt} = Ce^{-bt}, \text{ where } C = A_1 + A_2$$

In this equation, there is only one constant, hence it does not provide us the solution of differential equation (2) of second order.

Now, suppose $\sqrt{b^2 - \omega_0^2} = h$, which is a small quantity. Hence from eq. (11), we have

$$x = e^{-bt}[A_1 e^{ht} + A_2 e^{-ht}] = e^{-bt}[A_1(1 + ht + ...) + B(1 - ht + ...)]$$

Neglecting the small terms, containing h^2 and higher powers of h, we get

$$x = e^{-bt}[(A_1 + A_2) + h(A_1 - A_2)t] \text{ or } x = e^{-bt}(P + Qt) \quad ...(12)$$

where $P = A_1 + A_2$ and $Q = h(A_1 - A_2)$

Also $$\frac{dx}{dt} = e^{-bt} Q + e^{-bt}(-b)(P + Qt)$$

If initially at $t = 0$, the displacement of the particle is $x = x_0$ and there the velocity is v_0, then

$$x_0 = P \text{ and } v_0 = Q - bP$$

$= Q - bx_0$ or $Q = v_0 + bx_0$

$\therefore \quad x = [x_0 + (v_0 + bx_0)t]e^{-bt}$ ...(13)

This equation represents that initially the displacement increases due to the factor $[x_0 + (v_0 + bx_0)t]$. But as time elapses, the exponential term becomes relatively more important and the displacement decreases exponentially but quicker then overdamped case **[fig. 2].** In this case, the oscillatory motion does not just occur. Such a motion is called **critically damped** or **just aperiodic** or **just dead beat**. In a critically damped system, the displacement becomes zero in minimum time. The characteristic of critical damping is used in the measuring instruments, such an ammeters, voltmeters etc, where we want that the pointer or needle settles down to its final reading in a very short time.

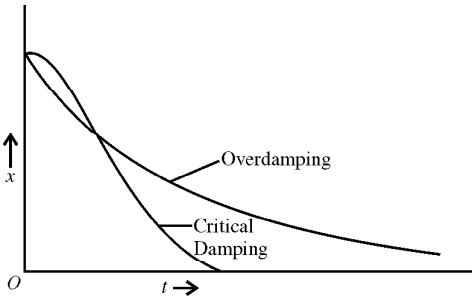


Fig. 2: Damped harmonic oscillator—overdamping and critical damping

3. Energy and Power Dissipation of Damped Harmonic Oscillator-relaxation Time

When a body oscillates in a medium (*e.g.*, oscillations of simple pendulum in air), the frictional or damping forces act on the body in a direction opposite to its movement. In this process work is done by the body in overcoming the frictional forces. This results in a continuous decrease in the mechanical energy of the oscillating body and thereby in dimnishing the amplitude of oscillation with time. Finally the vibrating body comes to rest, if left to itself for a sufficiently long time.

The total energy of a damped harmonic ocillator decreases with time. We are interested to find the rate at which the energy is dissipated i.e., the power dissipated, in the weak damping limit ($b << \omega_0$).

At any instant *t*, the displacement of a damped harmonic oscillator is given by

$$x = a_0 e^{-bt} \sin(\omega t + \phi) \quad ...(1)$$

where $a = a_0 e^{-bt}$ is the amplitude of damped harmonic oscillator at any time *t*.

In case of harmonic oscillator, we know that the total energy *E* of the oscillator is proportional to the square of the amplitude. When damping is low, we may consider that around time *t*, the oscillations are well described over some cycles by a simple harmonic motion, in which the total energy $E(t)$ is given by

$$E = \frac{1}{2} ka^2 = \frac{1}{2} ka_0^2 \, e^{-2bt} \quad ...(2)$$

Writing $E_0 = \frac{1}{2} ka_0^2$, the initial value of the energy, we have

$$E = E_0 \, e^{-2bt} \text{ or } E = E_0 \, e^{-t/\tau} \quad ...(3)$$

The energy of the oscillator after one period (T) is

$$E_T = E_0 \, e^{-2b(t+T)} = E \, e^{-2bT} \quad ...(4)$$

Hence the **average power dissipation** is given by

$$P_{av} = \frac{\text{Energy loss in one period}}{\text{Periodic time}} = \frac{E - E_T}{T} = \frac{E - Ee^{-2bT}}{T} = \frac{E - E(1-2bt)}{T} \text{ (as damping is small)}$$

$$\text{i.e.,} \quad P_{av} = 2bE = \frac{E}{\tau} \quad ...(5)$$

This dissipation of energy is caused by the damping force $\gamma \, dx/dt$. Hence *if we calculate the negative value of the average rate of doing work by this force*, i.e., $< \gamma \, (dx/dt)^2 >$, *then we will get exactly the same expression for power dissipation, as obtained above.* The proof is as follows:

Average rate of doing work by the damping force $= < F\frac{dx}{dt} > = - \gamma < \left(\frac{dx}{dt}\right)^2 >$

$$= -2bm\left(\frac{1}{2}a_0^2\omega^2 e^{-2bt}\right) = -2bE = -E/\tau \qquad ...(6)^*$$

For low damping ($b << \omega_0$), $< \left(\frac{dx}{dt}\right)^2 > = < \omega^2 a_0^{2e-2bt} \sin^2(\omega t + \phi) >$

$$= \frac{1}{2}\, a_0^2\omega^2 e^{-2bt} \qquad \left[\text{Time average of } \sin^2(\omega t + \phi = \frac{1}{2})\right]$$

where e^{-2bt} has been taken constant for a period.

Hence, **negative of the average rate of doing work** = $\frac{E}{\tau}$ = **power dissipation.**

This loss of energy in damped harmonic oscillations appears in the form of heat in the oscillating system and dissipated in the medium.

Relaxation Time τ

In case of damped harmonic oscillator, the total energy (E) at an instant t can be expressed as

$$E = E_0 e^{-t/\tau} \qquad ...(7)$$

where $\tau = 1/2b$ is relaxation time and b is damping constant.

When $t = \tau$,

$$E = E_0 e^{-1} \text{ or } E = \frac{E_0}{e} \qquad ...(8)$$

Thus *the relaxation time is defined as the time in which the total mechanical energy of the damped oscillator decreases to* 1/*e of its initial value.*

4. Quality Factor *Q*

When damping is small, the term Q, known as **quality factor**, is very widely used for various oscillating systems. The quality factor Q of an oscillating system is defined as the product of natural angular frequency (ω_0) (in absence of damping) and time constant (τ) of the oscillator i.e.,

$$Q = \omega_0\tau \qquad ...(1)$$

Evidently Q is a *dimensionless quantity.*

For a lightly damped harmonic oscillator, $\omega \approx \omega_0$. Therefore

$$Q = \omega\tau \qquad ...(2)$$

But, $$Q = \omega\tau = \frac{2\pi}{T}.\frac{E}{P_{av}} = \frac{2\pi E}{P_{av}T} \qquad \left[\because \omega = \frac{2\pi}{T} \text{ and } P_{av} = \frac{E}{\tau}\right]$$

or $$Q = \mathbf{2\pi}\frac{\textbf{average energy stored}}{\textbf{average energy loss per period}} \qquad ...(3)$$

Thus Q-factor of an oscillating system is equal to 2π times the ratio of the average energy stored to the average energy loss per period.

Now, for a mechanical oscillator, $\omega_0 = \sqrt{k/m}$ and relaxation time, $\tau = m/\gamma$, then from (1)

$$Q = \sqrt{\frac{K}{m}}.\frac{m}{\gamma} = \frac{\sqrt{mK}}{\gamma} \qquad ...(4)$$

where γ is damping coefficient.

Thus high value of Q implies low value of the damping for the oscillating system. This Q is a measure of the extent to which an oscillator is free from damping. In case of an **undamped oscillator**, $\gamma = 0$, and *hence Q is* **infinite**.

---

* For low damping ($b << \omega_0$), $< (dx/dt)^2 > = < a_0^2\omega^2 e^{-2bt} \sin^2(\omega\tau + \phi) > = \frac{1}{2}a_0^2\omega^2 e^{-2bt}$ where time average of $\sin^2(\omega t + \phi) = \frac{1}{2}$ and e^{-2bt} has been taken as constant for a time period.

The energy of the oscillation ($E = E_0 e^{-t/\tau}$) decreases to e^{-1} of its initial value in the time τ and in this time the oscillator performs $\omega_0\tau/2\pi = Q/2\pi$ oscillations.

5. *LCR* Circuit

Charge oscillation in an *LCR* circuit is an example of damped harmonic oscillator. Consider a coil of self inductance *L* and resistance *R*, connected to a capacitor *C* **[fig. 3]**. In the beginning, the condenser is charged to a charge q_0 on the right plate by a battery and then it is discharged through the inductance *L* and capacitance *C* by switching off the battery circuit and connecting the *LCR* circuit by a key. At any instant, if *q* be the charge on the capacitor *C*, then the potential difference across the capacitor *C* is

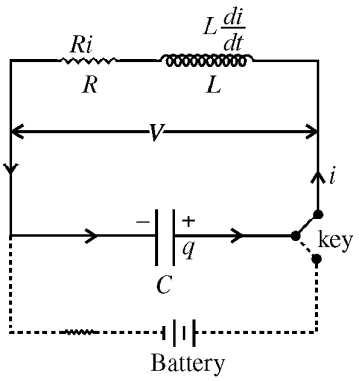


Fig. 3: LCR circuit.

$$V = \frac{q}{C} = L\frac{di}{dt} + Ri$$

or $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \qquad \left(\because i = -\frac{dq}{dt}\right) \qquad ...(1)$$

Here *L*, *R*, 1/*C* play the role of mass (*m*), damping (γ) and force constant (*K*) respectively.

Eq. (1) can be written as

$$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + \omega_0^2 q = 0 \qquad ...(2)$$

where $b = R/2L$ and $\omega_0^2 = 1/LC$.

Eq. (10) is exactly like the differential equation for the damped harmonic oscillator. The discharge will be oscillatory, when

$$\omega_0^2 > b^2 \quad \text{or} \quad \frac{1}{LC} > \frac{R^2}{4L^2}.$$

The solution of eq. (10) is of the form

$$q = q_0\, e^{-(R/2L)t} \sin(\omega t + \phi) \qquad ...(11)$$

We note that the charge amplitude, $q_0 e^{-(R/2L)t}$, decays exponentially with time. The decay in charge amplitude is caused by the resistance *R*, because in absence of *R* (i.e., $R = 0$), the charge amplitude will remain constant (q_0). Hence in a *LCR* circuit, resistance is only the dissipative element.

The frequency of the oscillatory discharge is given by

$$\nu = \frac{\omega}{2\pi} = \frac{\sqrt{\omega_0^2 - b^2}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad ...(12)$$

When $R = 0$, we obtain the case of harmonic charge oscillations and the frequency of oscillations, $\nu_0 = 1/2\pi\sqrt{LC}$ for *LC* circuit. Thus the presence of resistance in the circuit decreases the frequency.

***Q*-Factor in *LCR* circuit:** For low resistance circuit,

$$Q = \omega_0\tau = \frac{\omega_0}{2b} = \frac{L\omega_0}{R} \qquad ...(13)$$

For pure *LC* circuit $R = 0$, hence $Q = \infty$.

Now, if we differentiate eq. (11) with respect to *t* and place $-dq/dt = i$, then on arranging the differentiated terms, we get

$$i = i_0\, e^{-bt} \sin(\omega t + \theta) \qquad ...(14)$$

where i_0 is the maximum value of the circuit.

Cases when $\omega_0^2 \leq b^2$ or $\frac{1}{LC} \leq \frac{R^2}{4L^2}$

In case $\omega_0^2 < b^2$ or $\frac{1}{LC} < \frac{R^2}{4L^2}$, the circuit is overdamped and the discharge will be **non-oscillatory.**

Further, the case, when $\omega_0^2 = b^2$ or $\frac{1}{LC} = \frac{R^2}{4L^2}$ represents **critical damping** or **just aperiodic**.

Ex. 1. *Show that the amplitude of a damped harmonic oscillator falls to* $1/e$ *value in* 2τ *time, where* τ *is relaxation time.*

Sol. $a = a_0 e^{-bt} = a_0 e^{-t/2\tau}$ or $\frac{a}{a_0} = e^{-t}/2\tau$

According to the problem, $\frac{a}{a_0} = \frac{1}{e}$

Hence $\frac{1}{e} = e^{-t/2\tau}$ or $e^{-1} = e^{-t/2\tau}$ or $\frac{t}{2\tau} = 1, \therefore t = 2\tau.$

Ex. 2. *A particle of mass* 2 *gm is initially displaced through* 2 cm *and then released. The frictional force on it is* 5×10^{-3} N/m s^{-1}. *The restoring force on it is* 30×10^{-3} N/m. *Is it in oscillatory motion? If so, find period.* **(S.V.U. 2009)**

Sol. This is the case of damped harmonic oscillator.

The motion is oscillatory, when $b < \omega_e$. The time period is given by

$$T = \frac{2\pi}{\sqrt{(\omega_0^2 - b^2)}}. \qquad ...(i)$$

Here, $2b = \frac{\gamma}{m} = \frac{5 \times 10^{-3}}{2 \times 10^{-3}} = 2.5 \text{ sec}^{-1}$ or $b = 1.25 \text{ sec}^{-1}$

and $\omega_0 = \sqrt{\left(\frac{30 \times 10^{-3}}{2 \times 10^{-3}}\right)} = \sqrt{15} = 3.873 \text{ sec}^{-1}$

Therefore, $b < \omega_0$ and the motion is oscillatory. Substituting the values of b and ω_0 in (i), we get

$$T = \frac{2\pi}{\sqrt{15 - 1.5625}} = \mathbf{1.7132 \text{ sec.}}$$

Ex. 3. *A pendulum has a period of* 1 *second and amplitude of* 10°. *After* 10 *complete oscillations its amplitude reduces to* 5°. *Find the relaxation time.* **(K.U. 2012)**

Sol.: For damped simple harmonic oscillator, we have

$$a = a_0 e^{-bt}$$

After 10 complete oscillations, the time t is given by

$$t = 10 \times \text{period} = 10 \times 1 = 10 \text{ sec.}$$

Here, $a = 5°$ and $a_0 = 10°$

$$\therefore \quad 5 = 10e^{-10b} \quad \text{or} \quad \frac{1}{2} = e^{-10b} \quad \text{or} \quad e^{10b} = 2$$

Taking log on both sides: $10b \log_{10} e = \log_{10} 2$,

or $$10b = 2.303 \log_{10} 2 = 2.303 \times 0.3010$$

$$\therefore \quad b = \frac{\log_{10} 2}{10 \log_{10} e} = \frac{2.303 \times 0.3010}{10} = 0.06931$$

Therefore, relaxation time $\tau = \frac{1}{2b} = \frac{1}{2 \times 0.06931} =$ **7.214 sec**

Ex. 4. *The amplitude of vibration of a spring-mass system decreases from* 10 *cm, to* 2.5 *cm in* 200 *seconds. Find the damping constant and relaxation time. In how much time does the amplitude falls to* 1/*e times the initial value*?

If the ocillator performs 100 *oscillations in the given time, compare the periods with and without damping. What do you conclude*?

Sol. The amplitude at an instant t of a damped oscillator is given by

$$a = a_0 e^{-bt} \text{ or } e^{bt} = \frac{a_0}{a} \text{ or } b = \frac{1}{t} \log_e \frac{a_0}{a}$$

Here, $t = 200$ sec., $a_0 = 10$ cm and $a = 2.5$ cm,

$$\therefore \quad b = \frac{1}{200} \log_e \frac{10}{2.5} = \mathbf{6.93 \times 10^{-3}\ sec^{-1}}.; \ \tau = \frac{1}{2b} = \frac{1}{2 \times 6.93 \times 10^{-3}} = \mathbf{72\ sec.}$$

Now, for $\frac{a}{a_0} = \frac{1}{e}$, $\quad t = \frac{1}{b} \log_e e = \frac{1}{b} = \frac{1}{6.93 \times 10^{-3}} = \mathbf{144\ sec.}$

$$\text{Period, } T = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = \frac{200}{100} = 2 \text{ sec.}$$

Now, $$2 = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}}, \ \therefore \omega_0^2 = \pi^2 + b^2 = \pi^2 + (6.93 \times 10^{-3})^2 \text{ or } \omega_0 = 3.1416$$

$$\therefore \quad T_0 = 1.9999 \text{ sec} \simeq \mathbf{2\ sec}.$$

Note that the change in the time period due to damping is negligible.

Ex. 5. *A damped vibrating system starting from rest reaches a first amplitude of* 500 *mm, which reduces to* 50 *mm after* 100 *oscillations, each of period* 2.3 *seconds. Find the damping constant, relaxation time and correction for the first displacement for damping.*

Sol. The amplitude of damped vibrations is given by

$$a = a_0 e^{-bt}$$

Time period, $T = 2.3$ sec.

First amplitude of the system will be at $t = T/4$ and the amplitude after 100 oscillations *i.e*, 201th amplitude will occur at $(100T + T/4)$ time, hence

$$a_1 = a_0 e^{-bT/4} \text{ and } a_{201} = a_0 e^{-b(100T + T/4)};$$

$$\therefore \quad \frac{a_{201}}{a_1} = e^{-100bT} \text{ or } \frac{50}{500} = e^{-100 \times 2.3b} \text{ or } 10^{-1} = e^{-230b} \quad \text{or} \quad \log_e 10 = 230b,$$

$\therefore$ Damping constant, $b = 2.3/230 =$ **0.01 sec^{-1}**

$\therefore$ Relaxation time, $\tau = \frac{1}{2b} = \frac{1}{2 \times 0.01} =$ **50 sec.**

$\because \quad a_1 = a_0 e^{-bT/4}, \quad \therefore \quad 500 = a_0 e^{-0.01(2.3/4)}$

or $\quad a_0 = 500\ e^{0.023/4}$ or $\log_{10} a_0 = \log_{10} 500 + \frac{0.023}{4}\frac{1}{2.3}$

or $\quad \log_{10} a_0 = 2.699 + 0.0025 = 2.7015$, whence $a_0 = 502.9$ mm.

Thus in absence of damping, the amplitude of motion would have been 502.9 mm.

Hence the correction for the displacement for damping = 502·9 – 500 = **2·9 mm**.

Ex. 6. *An underdamped oscillator has its amplitude reduced to* $(1/10)^{th}$ *of its initial value after* 100 *oscillations. If the time period is* 2 seconds, *calculate* (1) *the damping constant and* (2) *the decay modulus.* **(A.U. 2009)**

Sol. The amplitude a of damped oscillator is given by

$$a = a_0 e^{-bt} \qquad ...(i)$$

where a_0 is the amplitude of undamped oscillator.

The first amplitude a_1 is obtained after $T/4$ sec, where T is the periodic time. Hence,

$$a_1 = a_0 e^{-bt(T/4)} \qquad ...(ii)$$

As the successive amplitudes occur at interval of $T/2$, the amplitude after 100 oscillations i.e., 201th amplitude will occur at time $\left(200\frac{T}{2}+\frac{T}{4}\right)$ or $\left(100T+\frac{T}{4}\right)$. Hence

$$a_{201} = a_0 e^{-b}(100T + T/4) \qquad ...(iii)$$

Dividing eq. (iii) by eq. (ii), we get

$$\frac{a_{201}}{a_1} = \frac{e^{-b(100T+T/4)}}{e^{-b(T/4)}} = e^{-100\ b\ T}$$

Given, $\frac{a_{201}}{a_1} = \frac{1}{10}$ and $T = 2$ sec.

$\therefore\ 10^{-1} = e^{-100 \times b \times 2}$ or $\log_e 10 = 200b$ or $2.3 \log_{10} 10 = 200b$, $\therefore\ b = \frac{2.3}{200} = \mathbf{0.0115 sec^{-1}}$ ...(i)

(ii) Decay modulus = $\frac{1}{b} = \frac{1}{0.0115} = \mathbf{86.8\ second.}$ ...(ii)

Ex. 7. *The differential equation for a certain system is*

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0$$

If $\omega_0 >> b$, *find the time in which*

(i) *amplitude falls to* $1/e$ *times the initial value.*

(ii) *energy of the system falls to* $1/e$ *times the initial value,*

(iii) *energy falls to* $1/e^4$ *times the initial value.*

Sol. (i) The given differential equation represents the equation of motion of damped harmonic oscillator, whose amplitude of vibration is given by

$$a = a_0 e^{-bt}$$

When amplitude falls to $1/e$ times the initial value a,

$$a_0 e^{-bt} = a_0/e \text{ or } e^{bt} = e^1 \quad \text{or} \quad bt = 1, \ \therefore \quad t = \mathbf{1/b\ sec}.$$

(ii) Energy, $E_t = E_0\ e^{-2bt} = E_0/e \quad \therefore \quad e^{2bt} = e$ or $2bt = 1$ or $t = \mathbf{1/2b\ sec}$.

(iii) $E_t = E_0\ e^{-2bt} = E_0/e^4$ or $2bt = 4$ or $t = \mathbf{2/b\ sec}$.

Ex. 8. *A damped harmonic oscillator has its first amplitude of* 20 *cm. It reduces to* 2 *cm after* 100 *oscillations, each of period* 4.6 *sec. Calculate the logarithmic decrement and damping constant. Find the number of oscillations in which the amplitude drops to* 50%.

Sol. $\lambda = \frac{1}{m}\log_e \frac{a_n}{a_{n+m}} = \frac{1}{100}\log_e \frac{20}{2} = \frac{1}{100}\log_e 10 = \frac{2.3}{100}\log_{10} 10 = \mathbf{2.3 \times 10^{-2}}$.

$\lambda = \frac{bT}{2}$, $\therefore$ $b = \frac{2\lambda}{T} = \frac{2\times 2.3\times 10^{-2}}{4.6} = \mathbf{10^{-2}\ sec^{-1}}$.

$m = \frac{1}{\lambda}\log_e \frac{a_n}{a_{n+m}} = \frac{1}{2.3\times 10^{-2}} \times 2.3 \times \log_{10} 2 = 10^2 \times 0.3 = \mathbf{30}$.

Ex. 9. *A mass of* 0.4 kg *is attached to a spring of force constant* 160 *N/m. If the relaxation time is* 0.05 sec, *find the Q factor.* **(K.U. 2012)**

Sol.: Quality factor $Q = \omega t$

Here, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{160}{0.4}} = 20$ and $\tau = 0.05$ sec.

$\therefore$ $Q = 20 \times 0.05 = \mathbf{1}$

Ex. 10. *The amplitude of an oscillator of frequency* 200 Hz *falls to* 1/10 *of its initial value after* 2000 *cycles. Calculate (i) its relaxation time, (ii) its quality factor, (iii) time in which its energy falls to* 1/10 *of initial value, (iv) damping constant.* **(A.N.U. 2011; V.S.U. 2014)**

Sol.: At an instant *t*, the amplitude of a damped harmonic oscillator is given by

$$a = a_0 e^{-bt}$$

The time of 2000 cycles $= \frac{2000}{200} = 10$ sec.

Thus in 2000 cycles or $t = 10$ sec., the amplitude a_0 falls to its $\frac{a_0}{10}$. Therefore,

$$\frac{a_0}{10} = a_0 e^{-b\times 10} \quad \text{or} \quad e^{10b} = 10$$

or $10b \log_{10} e = 1$ or $b = \frac{1}{10\log_{10} e} = \frac{2.3}{10} = 0.23\ \text{sec}^{-1}$

(i) Relaxation time $\tau = \frac{1}{2b} = \frac{1}{2\times 0.23} = \mathbf{2.174\ sec.}$

(ii) Quality factor $Q = \omega\tau = 2\pi\nu\tau = 2 \times 3.14 \times 200 \times 2.174 = \mathbf{2730}$

(iii) Energy $E = E_0 e^{-t/\tau}$

For $E = \frac{E}{10}$, $\frac{E_0}{10} = E_0 e^{-t/\tau}$ or $\frac{1}{10} = e^{-1t/\tau}$ or $e^{t/\tau} = 10$

$\therefore$ $t = \tau \log_e 10 = 2.174 \times 2.3 = \mathbf{5\ sec.}$

(iv) Damping constant $b = \mathbf{0.23\ sec^{-1}.}$

Ex. 11. *The frequency of a tuning fork is* 300 *Hz. If its quality factor Q is* 5×10^4, *find the time after which its energy becomes* 1/10 *of its initial value. What is the value of relaxation time for the oscillator*? **(O.U. 2014, 09; S.V.U. 2006)**

Sol. $E = E_0 e^{-2bt} = E_0 e^{-t/\tau}$. When energy becomes 1/10 of its initial value,

$E_0/10 = E_0 e^{-t/\tau}$ or $10 = e^{t/\tau}$ or $t/\tau = \log_e 10 = 2.3$, $\therefore$ $t = 2.3\tau$

But $\tau = \frac{Q}{\omega} = \frac{Q}{2\pi n} = \frac{5\times 10^4}{2\times 3.14\times 300}$, $\therefore t = \frac{2.3\times 5\times 10^4}{2\times 3.14\times 300} = \mathbf{61.1\ sec}$.

Ex. 12. *The amplitude of a second pendulum falls to half initial value in* 150 sec. *Calculate the Q-factor.* **(A.N.U. 2013; S.K.U. 2006)**

Sol. The amplitude of second pendulum in the presence of damping forces will decrease according to the relation:

$$a = a_0 e^{-bt} \text{ or, } \frac{a}{a_0} = e^{-bt}$$

Here, $\frac{a}{a_0} = \frac{1}{2}$ and t = 150 sec.

$$\therefore \quad \frac{1}{2} = e^{-150b} \text{ or } e^{150b} = 2 \text{ or } 150b = \log_e 2 = 2.303 \log_{10} 2 \text{ or } 150b = 0.6932$$

$$\therefore \quad b = \frac{0.6932}{150} = 4.62 \times 10^{-3} \text{ sec}^{-1}$$

For second pendulum, T = 2 sec, $\therefore\ \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi = 3.14$ rad/sec.

Also $$\tau = \frac{1}{2b} = \frac{1}{2 \times 4.62 \times 10^{-3}}$$

Hence, $$Q = \omega\tau = \frac{3.14}{2 \times 4.62 \times 10^{-3}} = \mathbf{339.8}$$

Ex. 13. *The Q-factor of a tuning fork of frequency* 512 Hz *is given as* 8×10^4. *Calculate the time in which the energy is reduced by* $\left(1/e\right)^{\text{th}}$ *of its initial value.* **(S.V.U. 2014; A.N.U. March 2007)**

Sol. $$Q = \omega\tau, \ \therefore\ \tau = \frac{Q}{\omega} = \frac{8 \times 10^4}{2 \times 3.14 \times 512} = 24.88 \text{ sec.}$$

Now, $$E = E_0 e^{-t/\tau} \text{ or } \frac{E}{E_0} = e^{-t/\tau}, \ \therefore\ \frac{1}{e} = e^{-\frac{t}{24 \cdot 88}} \text{ or } e^{-1} = e^{-\frac{t}{24 \cdot 88}}$$

$$\therefore \quad \frac{t}{24.88} = 1 \text{ or } t = \mathbf{24.88 \text{ sec}}$$

Ex. 14. *The Q-factor of an oscillator is* 500. *Find its initial energy if its amplitude is* 0.01 m. *Also calculate the energy lost in first cycle. Given spring constant* $S = m\omega^2$ = 100 N/m.

Sol. Initial energy $E = \frac{1}{2} m\omega^2 a^2 = \frac{1}{2} Sa^2 = \frac{1}{2} \times 100 \times (0.01)^2 = 5 \times 10^{-3}$ joule

$$Q = 2\pi\left(\frac{\text{Energy stored}}{\text{Energy lost in period}}\right)$$

Therefore, $$500 = 2 \times 3.14 \times \frac{5 \times 10^{-3}}{E_i}$$

whence $$E_l = \frac{2 \times 3.14 \times (5 \times 10^{-3})}{500} = \mathbf{6.28 \times 10^{-5} \text{ joule}}$$

Ex. 15. *A mass of* 0.3 kg *is tied to a spring. Its frequency is* 2 Hz *and Q factor is* 60. *Find spring constant, damped factor and mechanical resistance.* **(K.U. 2013, 12)**

Sol. Quality factor $Q = \omega\tau$

Here, $\omega = 2\pi\nu = 2\pi \times 2 \text{ sec}^{-1}$

$\therefore \quad \tau = \frac{Q}{\omega} = \frac{60}{2\pi \times 2} = \frac{15}{\pi}$ sec.

Also, $\omega^2 = \frac{k}{m}$, $\therefore \; k = \omega^2 m = (4\pi)^2 \times 0.3 = 16 \times (3.14)^2 \times 0.3 =$ **47.33 N/m**

Now, $\tau = \frac{1}{2b} = \frac{m}{\gamma}$ or $\gamma = \frac{m}{\tau} = \frac{0.3}{15} \times \pi =$ **0.0628 kg/sec.**

This γ is the mechamical resistance.

Damping factor (constant) = $b = \frac{\gamma}{2m} = \frac{0.0628}{2 \times 0.3} =$ **0.105 sec^{-1}**

Ex. 16. *The quality factor of a sonometer wire is* 2×10^3. *On plucking it makes* 240 *vibrations per second. Calculate the time in which amplitude decreases to half the intitial value.* ($\log_{10} 2 = 0.3010$)

Sol. Quality factor, $Q = \omega\tau$

Here, $Q = 2 \times 10^3$ and $\omega = 2\pi\nu a = 2\pi \times 240$ rad/sec.

$\therefore \quad 2 \times 10^3 = 2 \times 3.14 \times 240 \times \tau$

whence, $\tau = \frac{2 \times 10^3}{2 \times 3.14 \times 240} = 1.327$ sec.

The amplitude at an instant t is given by

$$a = a_0 e^{-bt} = a_0 e^{-t}/2\tau \qquad \left(\because b = \frac{1}{2\tau}\right)$$

Given, $\frac{a}{a_0} = \frac{1}{2}$. Hence

$$\frac{a}{a_0} = \frac{1}{2} = e^{-\frac{t}{2\tau}} = e^{-\frac{t}{2 \times 1 \cdot 327}}$$

$\therefore \quad e^{t/(2 \times 1.327)} = 2$ or $\frac{t}{2 \times 1.327} = \log_e 2 = 2.3 \times \log_{10} 2$

whence $t = 2 \times 1.327 \times 2.3 \times 0.3010 =$ **1.84 sec.**

Ex. 17. *The Q value of a spring loaded with* 0.3 kg *is* 60. *It vibrates with a frequency of* 2 Hz. *Calculate the force constant and mechanical resistance.* **(A.N.U. 1997)**

Sol. Spring force constant $k = m\omega^2$

Here, $m = 0.3$ kg and $\omega = 2\pi\nu = 2\pi \times 2 = 4\pi$ rad/sec.

$\therefore \quad k = 0.3 \times (4\pi)^2 = 0.3 \times (4 \times 3.14)^2 =$ **47.37 N/m.**

Now, $Q = \omega\tau = \frac{\omega}{2b} = \frac{\omega m}{\gamma}$

whence, $\gamma = \frac{\omega m}{Q} = \frac{4\pi \times 0.3}{60}$ $(\because Q = 60)$

$=$ **0.0628 kg/sec.**

Ex. 18. *A box of* 100 *gm is attached to one end of a spring whose other end is fixed to a rigid support. When a mass of* 900 *gm is placed inside the box the system performs* 4 *vibrations per second and the amplitude falls from* 2 *cm to* 1 *cm in* 15 *seconds. Calculate* (i) *the force constant,* (ii) *the relaxation time,* (iii) *Q of the system and* (iv) *Q when only* 100 *gm* (*empty box*) *is with the spring.* ($\log_e 10 = 2.3$)

Sol. (i) Frequency of vibration of the system, $\nu = 4$ sec^{-1},

$\therefore \quad \omega_0 = 2\pi\nu = 2 \times 3.14 \times 4 = 25$ rad./sec. (if the damping is small)

But, $\omega_0 = \sqrt{k/m}$ or $c = m\,\omega_0^2 = 1 \times 25^2 =$ **625 N-m**.

($\because$ m = 100 gm + 900 gm = 1000 gm = 1 kg)

(ii) At any instant t, the amplitude of the system is given by

$$a = a_0\, e^{-bt} \text{ or } 0.01 = 0.02\, e^{-15b} \text{ or } \frac{1}{2} = e^{-15b} \text{ or } 15b = \log_e 2$$

$$\therefore \quad b = \frac{\log_e 2}{15} = \frac{2.3 \times 0.3}{15} = \textbf{0.046 sec}^{-1}$$

$$\therefore \quad \text{Relaxation time, } \tau = \frac{1}{2b} = \frac{1}{2 \times 0.046} = \textbf{11 sec}.$$

(iii) $Q = \omega_0 \tau = 25 \times 11 =$ **275**.

Damping force = $\gamma\, dx/dt$, where $\gamma = 2\,bm = 2 \times 0.046 \times 1$.

(iv) When the box is empty, i.e., the mass is 100 gm = 0·1 kg, the damping force $\gamma\, dx/dt$ and restoring force kx remain unchanged. Now,

$$\omega_0 = \sqrt{\frac{k}{m'}} = \left(\frac{625}{0.1}\right)^{\frac{1}{2}} = 79 \text{ rad./sec. and } \tau = \frac{1}{2b} = \frac{m'}{\gamma} = \frac{0.1}{2 \times 0.046 \times 1} = 1.1 \text{ sec.}$$

$$\therefore \quad Q = \omega_0 \tau = 79 \times 1.1 = \textbf{87}.$$

Ex. 19. *A condenser of capacity* 20 μF *is charged and then discharged through an inductance of* 10 mH. *Calculate the frequency of resulting oscillations.* (*Resistance of the coil can be neglected*). **(N.U. 2008)**

Sol. Frequency $\nu = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

When $R = 0$, $\nu = \frac{1}{2\pi\sqrt{LC}}$

Here, L = 10 mH = 10 × 10^{-3} H and C = 20 μF = 20 × 10^{-6} F.

$$\therefore \quad \nu = \frac{1}{2 \times 3.14} \sqrt{10 \times 10^{-3} \times 20 \times 10^{-6}} = \textbf{356.1 Hz.}$$

Ex. 20. *Calculate the frequency and Q-factor for a circuit with* L = 2 mH, C = 5 μF and R = 0.2 Ω. **(A.N.U. 2002)**

Sol. Frequency $\nu = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2\pi} \sqrt{\frac{1}{2 \times 10^{-3} \times 5 \times 10^{-6}} - \frac{(0.2)^2}{4 \times (2 \times 10^{-3})^2}}$

$$= \frac{1}{2\pi\sqrt{2 \times 10^{-3} \times 5 \times 10^{-6}}} = \textbf{1592 Hz}$$

$$\therefore \quad Q = \frac{\omega L}{R} = \frac{2\pi\nu L}{R} = \frac{2 \times 3.14 \times 1592 \times 2 \times 10^{-3}}{0.2} = \textbf{99.27}$$

Ex. 21. *A condenser of* 2 *microfarad capacity is discharged through a* 1 *ohm resistance and* 2 *henry inductance. Show that the system will act as damped harmonic oscillator. Calculate the frequency and quality factor of LC circuit and find the time in which the amplitude of oscillations is reduced to* 1%.

Sol. $\nu = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

Here, $C = 2 \times 10^{-6}$ farad, L = 2 henry and R = 1 ohm.

$$\therefore \quad \nu = \frac{1}{2\pi} \sqrt{\frac{1}{2 \times 2 \times 10^{-6}} - \frac{1}{16}} = \frac{10^3}{2\pi \times 2} = \textbf{79.5 Hz}.$$

As resistance is low, so

$$Q = \frac{L\omega}{R} = \frac{2\times 2\pi}{1} \times \frac{10^3}{2\pi \times 2} = \mathbf{10^3}$$

The amplitude, $q = q_0 e^{-R/2L}$

As amplitude decreases to 1% of its initial value q_0, we have

$$\frac{q_0}{100} = q_0 e^{-t/4} \text{ or } e^{t/4} = 100 \text{ or } \frac{t}{4} = \log_e 100 = 2.3 \times 2 \text{ or } t = \mathbf{18.4\ sec}.$$

Ex. 22. *In a circuit, L = 2 mH and C = 5 μF. If R = 1 Ω, 40 Ω, and 100 Ω, discuss the behaviour of the circuit in each case. Find the frequency of oscillation and Q-factor in the relevant cases.*

Sol. Here, $L = 2 \times 10^{-3}$ H and $C = 5 \times 10^{-6}$ F

$$\therefore \quad \frac{1}{LC} = \frac{1}{2\times 10^{-3} \times 5 \times 10^{-6}} = 10^8$$

Case I: $R = 1\ \Omega$, $\quad \therefore \quad \frac{R^2}{4L^2} = \frac{1}{4\times (2\times 10^{-3})^2} = 6.25 \times 10^4$

$\therefore \frac{1}{LC} > \frac{R^2}{4L^2}$ and hence the **circuit is oscillatory.**

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \mathbf{1.59\ KHz}.$$

and $$Q = \frac{\omega_0 L}{R} = \frac{2\pi \times 1.59 \times 10^3 \times 2 \times 10^{-3}}{1} = \mathbf{19.98}$$

Case II: $R = 40\ \Omega$, $\therefore \quad \frac{R^2}{4L^2} = \frac{40\times 40}{4\times (2\times 10^{-3})^2} = 10^8$

Hence $$\frac{1}{LC} = \frac{R^2}{4L^2} \quad (\text{or } \omega_0^2 = b^2)$$

This is the case of **cirtical damping.**

Case III: $R = 100\ \Omega$, $\therefore \quad \frac{R^2}{4L^2} = \frac{100^2}{4\times (2\times 10^{-3})^2} = 6.25 \times 10^8$

Therefore $$\frac{R^2}{4L^2} > \frac{1}{LC} \quad (\text{or } b^2 > \omega_0^2)$$

This is the case of **overdamping.**

6. Forced (or Driven) Harmonic Oscillator

If an external periodic force is applied on a damped harmonic oscillator, then the oscillating system is called **driven** *or* **forced oscillator** *and its oscillations are called* **forced** *(or* **driven***)* **vibrations.** *The applied periodic force is called* **driver.** *Let the periodic force represented by* $F = F_0 \sin pt$, *where* F_0 *is the maximum value of the force of frequency* $p/2\pi$. Hence the differential equation of motion is

$$m\frac{d^2x}{dt^2} = -\gamma \frac{dx}{dt} - kx + F$$

where m is the mass of the ocsillating particle, $-\gamma\, dx/dt$ is the damping force and $-kx$ is the restoring force. Now, the above equation can be written as

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m} \sin pt$$

or $$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt \qquad ...(1)$$

where $2b = \gamma/m = 1/\tau$, $\omega_0 = \sqrt{k/m} = 2\pi\nu_0$ = the natural angular frequency of the system in the absence of damping and driven force, and $f_0 = F_0/m$.

Eq. (1) represents the ***differential equation of motion for the driven harmonic oscillator***.

If we apply a harmonically oscillating force to a damped harmonic oscillator at a frequency (p), which is, in general, different to the natural frequency of the oscillator, in the beginning we obtain the transient response i.e., in the initial stages, the oscillator tends to vibrate with its natural frequency (ω), while the impressed periodic force tries to impose its own frequency (say p) upon it and hence a sort of tussle takes place between the two. When the transient effects have died away, the oscillator ultimately acquires the steady state and vibrates at the frequency (p) of the applied harmonic force and not at its natural frequency (ω). Such oscillations of the system at the frequency of the applied periodic force are called **forced** or **driven oscillations.** The external applied harmonic force is called the **driver** and the oscillating system itself is called the **driven** or **forced oscillator**. The driving harmonic force delivers periodic impulses to the oscillator so that the loss of energy in doing work against the damping forces is recovered. Consequently in the steady state, the oscillator vibrates with constant amplitude at the frequency of applied harmonic force.

Examples of driven harmonic oscillator:

1. Vibration of a bridge under the rhythm of marching soliders over it.
2. The oscilltions of sonometer wire when a vibrating fork is placed on its sounding box.
3. Charge oscillations in an *LCR* circuit fed by alternating voltage.
4. Oscillations of a stretched wire carrying alternating current between the poles of a magnet.

Solution of the Differential Equation of the Driven Harmonic Oscillator–Steady State Solution

Let the steady state solution of the driven harmonic oscillator be

$$x = a \sin (pt - \theta) \qquad ...(2)$$

because the amplitude (a) of the forced oscillations remains constant and the frequency of vibrations is equal to the ($p/2\pi$) of the force. Here θ represents the phase difference between the force and the resultant displacement of the system. Now, from eq. (3), we have

$$\frac{dx}{dt} = pa \cos (pt - \theta) \text{ and } \frac{d^2x}{dt^2} = -p^2 a \sin (pt - \theta)$$

Substituting these values in eq. (1), we get

$$-p^2a \sin (pt - \theta) + 2bpa \cos (pt - \theta) + \omega_0^2 a \sin (pt - \theta) = f_0 \sin (pt - \theta + \theta)$$

or $$a(\omega_0^2 - p^2) \sin (pt - \theta) + 2bpa \cos (pt - \theta) = f_0 \cos \theta \sin (pt - \theta) + f_0 \sin \theta \cos (pt - \theta) \qquad ...(3)$$

This relation holds for all the values of t. Hence when $(pt - \theta) = 0$ or $2m\pi$, then from eq. (3), we have

$$2bpa = f_0 \sin \theta \qquad ...(4)$$

and when $(pt - \theta) = \pi/2$ or $(2m + 1)\pi/2$, then

$$(\omega_0^2 - p^2)a = f_0 \cos \theta \qquad ...(5)$$

Squaring eqs. (4) and (5) and on adding, we get

$$4b^2p^2a^2 + (\omega_0^2 - p^2)^2a^2 = f_0^2 (\sin^2\theta + \cos^2\theta) \text{ or } [4b^2p^2 + (\omega_0^2 - p^2)]a^2 = f_0^2$$

$$\therefore \quad a = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}} = \frac{F_0/m}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}} \qquad ...(6)$$

Dividing eq. (4) by eq. (5), we get

$$\tan \theta = \frac{2bp}{\omega_0^2 - p^2} \text{ or } \theta = \tan^{-1}\left(\frac{2bp}{\omega_0^2 - p^2}\right) \qquad ...(7)$$

Substituting these values in eq. (2), we get

$$x = \frac{F_0/m}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}} \sin(pt - \theta) \quad ...(8)$$

where θ is given by eq. (7)

Thus the amplitude of forced oscillations not only depends on force amplitude (F_0) and mass of the oscillating body (m) but also on the driving frequency (p), natural frequency (ω_0) of the oscillator (in absence of damping) and the damping constant (b). The phase of the displacement lags that of the harmonic force by angle θ which is also depending on p, ω_0 and b. The reader may note that we may sometime use word frequency for angular frequency. Further note that the steady state solution has the frequency of the driving force and its amplitude is constant.

In case of undamped system, $b = 0$ so that from eq. (7), $\theta = 0$ and therefore from eq. (8)

$$x = \frac{F_0/m}{\omega_0^2 - p^2} \sin pt \quad ...(9)$$

Thus *for a simple harmonic oscillator,* when a periodic force $F = F_0 \sin pt$ is applied, *the driven force and displacemet* (x) *are in the same phase* (as $\theta = 0$). Hence we conclude that **phase lag is essentially a consequence of damping.**

Variation of Amplitude (a) **and Phase** (θ) **with the Driving Frequency** (p)

We consider below the amplitude (a) and phase lag (θ) of the oscillator at different driving frequencies in the case of low damping.

Case I: Low driving frequency ($p << \omega_0$): When the driving frequency (p) is much less than the natural frequency (ω_0) of the oscillator (in absence of damping), we obtain from eq. (6)

$$a \to \frac{f_0}{\omega_0^2} = \frac{F_0/m}{k/m} = \frac{F_0}{k} \text{ and } \tan\theta \to 0 \text{ or } \theta \to 0 \quad ...(10)$$

Thus at the low driving frequency, the amplitude does not depend on the mass (m) of the oscillating system or damping but it is controlled by the spring factor (k). In this limit, the periodic force displaces slowly the system against the restoring force of the spring and the displacement is said to be in phase with the driving force.

Case II: Resonant response: Now, *if the driving frequency is increased*, the value of the amplitude a increases. To find the condition for maximum amplitude (response), we differentiate eq. (6) with respect to p and put $da/dp = 0$ i.e.,

$$\frac{da}{dp} = \left(\frac{F_0}{m}\right)\left(-\frac{1}{2}\right)\frac{2(\omega_0^2 - p^2)(-2p) + 8b^2p}{[(\omega_0^2 - p^2)^2 + 4b^2p^2]^{\frac{3}{2}}} = 0 \text{ or } 2b^2 - \omega_0^2 + p^2 = 0$$

Hence $$p = \sqrt{\omega_0^2 - 2b^2} = p_r \text{ (say)} \quad ...(11)$$

For this value of p, d^2a/dp^2 is found to be negative. Hence the amplitude a is maximum at $p = p_r$, given by eq. (11).

At $p_r = \sqrt{\omega_0^2 - 2b^2}$, a is maximum, say a_{max} amplitude, and from eq. (6),

$$a_{max} = \frac{f_0}{2b\sqrt{\omega_0^2 - b^2}} \quad ...(12)$$

Thus the amplitude of the oscillating system is maximum at a particular value of the driving frequency (p_r).

This phenomenon in which the amplitude of the driven oscillator is maximum at a certain driving frequency (p_r) *is called* **amplitude resonance** *and this frequency is called as* **amplitude.**

Resonant frequency: The value of this angular frequency ($p_r = \sqrt{\omega_0^2 - 2b^2}$) is less than the natural frequency of undamped or damped vibrations (ω_0 or $\omega = \sqrt{\omega_0^2 - b^2}$) of the oscillator. However, we are considering low damping so that $p_r \approx \omega_0$ and then

$$a_{max} = \frac{f_0}{2b\omega_0} = \frac{f_0\tau}{\omega_0} \quad ...(13)$$

Thus in case of zero damping ($b = 0$), the amplitude will be infinity. But this is not possible as the frictional forces acting on the oscillating system, are never zero.

For low damping, the ratio of the response (amplitude) at resonance to the response at low or zero driving frequency is given by

$$\frac{\textbf{Response (amplitude) at resonance (for low damping)}}{\textbf{Response (amplitude) at zero driving frequency}} = \frac{f_0\tau/\omega_0}{f_0/\omega_0^2} = \omega_0\tau = Q \quad ...(14)$$

Thus *the response at resonance is high for a system with high Q-factor.*

At $p = \omega_0$, we obtain from eq. (7)

$$\tan\theta = \infty \text{ or } \theta = \frac{\pi}{2} \text{ and } a = \frac{f_0}{2b\omega_0} \quad ...(15)$$

Thus at $p = \omega_0$, the amplitude is less than a_{max} ($= f_0/2b\sqrt{\omega_0^2 - b^2}$).

However for low damping, $p_r \approx \omega_0$ and then amplitude is maximum, given by eq. (15).

Case III: High driving frequency ($p >> \omega_0$): When the driving frequency is much high,

$$\tan\theta \to -\frac{2b}{p} \to -0 \text{ or } \theta \to \pi$$

$$\text{and } a = \frac{f_0}{p^2} \text{ nearly or } a = \frac{F_0}{mp^2} \quad ...(16)$$

Thus in the high frequency limit, the response decreases rapidly as $1/p^2$ and tends to zero. In this limit, the oscillating mass responds like a free object, shaking rapidly in opposite phase ($\theta = \pi$) to the impressed force of high frequency.

The variation of amplitude a with the frequency p of the driving force is shown in **fig. 4(a)**. As the angular frequency is increased from the low or zero frequency limit, the amplitude a increases continuously and becomes maximum at a frequency p_r, which is just before and close to ω_0. This is **amplitude resonance.** Now, if the driving frequency is further increased, the amplitude (a) decreases regularly and at very high frequencies approaches to very small value. When the damping is greater, the resonant frequency (p_r) is less than ω_0 but in case of low damping the resonant frequency of the force is nearly equal to the natural frequency (ω_0) of the oscillator. It is also clear from these curves that for low damping the height of the peak becomes greater and when the damping is zero (in the ideal case) the height of the peak (i.e., the maximum amplitude) rises to infinity.

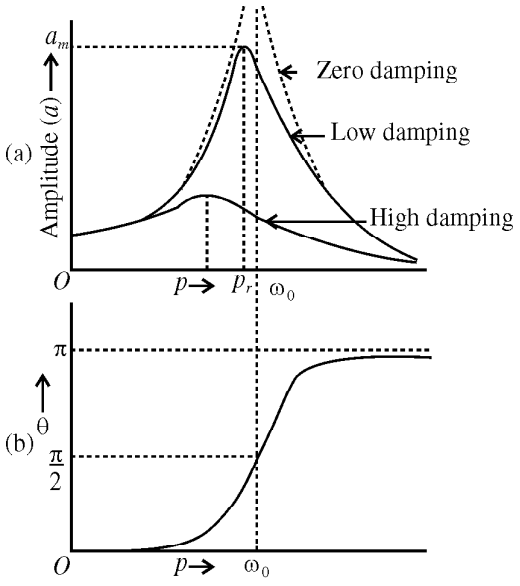


Fig. 4: Forced oscillations: (a) Variation of amplitude (a) of the oscillating system with the driving frequency; (b) Phase lag (θ) of displacement relation to the driving force as a function of driving frequency (p).

Regarding the phase, we note from our analysis that the displacement x always lags in phase θ relative to the driving force; the value of θ is zero at low frequencies, increases with frequency p and passes $\pi/2$ at $p = \omega_0$ and finally becomes π at high frequencies **[fig. 4(b)]**.

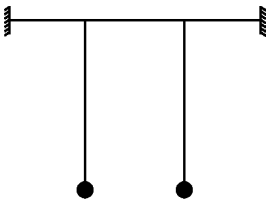


Fig. 5.

Examples of Resonant Vibration

1. When two strings of the same frequency are stretched on the same board and one of them is set into vibrations, it is observed that the other string starts to vibrate.

2. If two pendulums of the same periodic time are suspended from a thin cord, and one of the pendulum is oscillated, the other pendulum begins to oscillate **[fig. 5].**

3. Soldiers passing over a suspension bridge always break steps because if the frequency of their march becomes equal to the natural frequency, then due to resonance the bridge may set into violent vibrations and collapse.

4. In the determination of the speed of sound in air in the laboratory by resonance tube method, a loud sound is heard when the frequency of the vibrating fork becomes equal to the frequency of the air column of the tube.

5. In the driven *LCR* circuit, the current becomes large when the frequency of the applied alternating voltage becomes equal to the resonant frequency of the circuit.

7. Resonance (Amplitude) and Sharpness of Resonance – Half Width of Resonance Curve

In case of driven harmonic oscillator, if we study the variation of amplitude of the oscillating system with the driving frequency, we find that at a particular frequency, called the resonance frequency (p_r), the amplitude is maximum. This phemomenon is called **amplitude rsonance.** The amplitude resonance occurs at the frequency, given by

$$p_r = \sqrt{\omega_0^2 - 2b^2} \qquad ...(1)$$

where ω_0 is frequency of undamped oscillator and b is damping constant.

At this frequency, the amplitude of the oscillating system is given by

$$a_{\max} = \frac{f_0}{2b\sqrt{\omega_0^2 - b^2}} = \frac{F_0/m}{2b\sqrt{\omega_0^2 - b^2}} \qquad ...(2)$$

where F_0 is the amplitude of the periodic force $F = F_0 \sin pt$ and m is the mass of the oscillator.

If the damping is small $b << \omega_0$, then the resonant frequency is equal to the frequency of the oscillator i.e.,

$$p_r = \omega_0 \text{ and } a_{\max} = \frac{F_0}{2bm\omega_0} = \frac{F_0\tau}{m\omega_0} \qquad ...(3)$$

However at zero or low driving frequency, the amplitude is given by

$$a_0 = \frac{f}{\omega_0^2} = \frac{F_0}{m\omega_0^2} \qquad ...(4)$$

Thus,
$$\frac{a_{\max}}{a_0} = \frac{F_0\tau/m\omega_0}{F_0/m\omega_0^2} = \omega_0\tau = Q$$

i.e., Quality factor, $Q = \dfrac{a_{\max}}{a_0} = \dfrac{\text{Amplitude at resonance}}{\text{Amplitude at zero (or low) driven frequency}}$ ...(5)

Obviously the amplitude (response) will be more for a high Q-oscillating system.

The amplitude of the oscillating system decreases from the maximum value $A_{\max}$ with the change

(decrease or increase) of the frequency of the impressed force. In **fig 4(a)**, for different values of damping, curves have been drawn between the driven frequency and the amplitude.

Sharpness of resonance: *The term* **sharpness of resonance** *refers to the rate of fall in amplitude with the change of force frequency on each side of the resonant frequency.* When the damping is low, the response (amplitude) falls of very rapidly on either side of the resonant frequency and we say that the resonance is **sharp**. On the other hand, for high damping the response falls off very slowly on either side of the resonant frequency and the resonance is said to be **flat**. Thus, the resonance is flat or sharp according to the damping for the oscillating system is large or small. Familiar examples of a flat or sharp resonance are the resonance of an air column and a sonometer wire with a tuning fork respectively. Due to its large damping the air column responds with tuning fork over a fairly wide range in the neighbourhood of resonance. Thus in this case it is actually difficult to get an exact point of resonance and resonance is said to be **flat.** But in case of sonometer wire, the damping is small so that the resonance is **sharp** and hence to have the resonant point, the length of wire is adjusted very precisely.

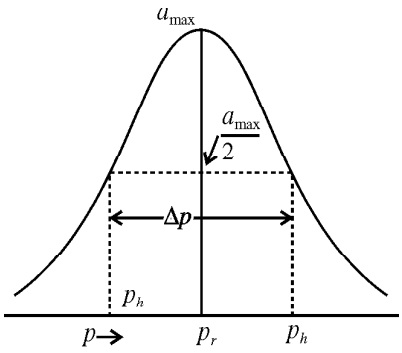


Fig. 6: Amplitude resonance curve and half width (Δp).

Half width of amplitude resonance curve: If p_h is the value of angular frequency when the amplitude falls to half the value of p_r at resonance, then Δp,the change in p, is called **half width** of the resonance curve **[fig. 6],** i.e.,

$$\text{Half width, } \Delta p = |p_h - p_r| \qquad ...(6)$$

$$\text{At resonance, } a_{max} = \frac{f_0}{2b\sqrt{\omega_0^2 - b^2}} = \frac{f_0}{\sqrt{4b^2\omega_0^2 - 4b^4}} \qquad ...(7)$$

At the frequency p_h, the amplitude is $a_{max}/2$.

$$\therefore \quad \frac{a_{max}}{2} = \frac{f_0}{\sqrt{(\omega_0^2 - 2b^2 - p_h^2)^2 + 4b^2\omega_0^2 - 4b^4}} \quad \text{[using eq. (6), Art. 8.6]} \qquad ...(8)$$

From (7) and (8), we have

$$(\omega_0^2 - 2b^2 - p_h^2)^2 + 4b^2\omega_0^2 - 4b^2 = 4(4b^2\omega_0^2 - 4b^4)$$

But, $$p_r^2 = \omega_0^2 - 2b^2$$

$$\therefore \quad (p_r^2 - p_h^2)^2 = 3(4b^2\omega_0^2 - 4b^4) = 3(4b^2p_r^2 + 4b^4) \qquad ...(9)$$

If damping is small, $4b^4$ is negligible and then we have

$$p_r^2 - p_h^2 = \pm\sqrt{3}\,(2bp_r) \text{ or } p_h^2 = p_r^2 \mp \sqrt{3}\,(2bp_r)$$

or $$p_h = p_r\left(1 \mp \sqrt{3}\,\frac{2b}{p_r}\right)^{1/2} = p_r\left(1 \mp \frac{\sqrt{3}b}{p_r}\right) = p_r \mp \sqrt{3}b$$

$$\therefore \quad \textbf{Half width, } \Delta p = |p_h - p_r| = \sqrt{3}b \qquad ...(10)$$

This is the expression for half width of the amplitude resonance curve.

$$\textbf{Full width} = 2\Delta p = 2\sqrt{3}b \qquad ...(11)$$

But, $$Q = \omega_0\tau = \frac{\omega_0}{2b} \text{ or } b = \frac{\omega_0}{2Q}$$

hence Half width, $\Delta p = \sqrt{3}\,\dfrac{\omega_0}{2Q}$ ...(12)

Thus ***for high Q-oscillating system the half width of the amplitude resonance curve is small***.

8. Quality Factor and Effect of Damping

The quality factor of an oscillating system with damping is defined as

$$Q = \omega_0\tau = \frac{\omega_0}{2b} \qquad ...(1)$$

where τ is relaxation time and b is damping constant.

The quality factor can be interpreted in terms of amplitude of the oscillating system because [eq. (14), Art. 6].

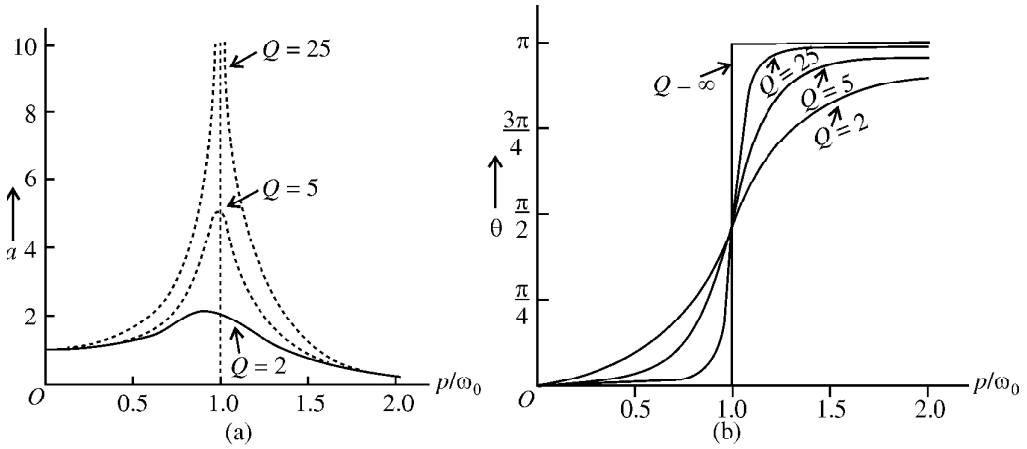


Fig. 7: For different Q (a) Variation of amplitude (a) with driving frequency (ω); (b) Variation of phase lag (θ) with driving frquency (ω).

$$\frac{\text{The amplitude at resonance (for low damping)}}{\text{The at amplitude at low frequency of the driving force } (p \to 0)} = \frac{f_0\tau/\omega_0}{f_0/\omega_0^2} = \omega_0\tau = Q \qquad ...(2)$$

Thus the *Q*-factor can be defined as the ratio of amplitude at resonance to the amplitude at low frequencies.

In case of driven oscillator,

$$a = \frac{F_0/m}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}} \text{ and } \theta = \tan^{-1}\left(\frac{2bp}{\omega_0^2 - p^2}\right) \qquad ...(3)$$

Let, $a_0 = \dfrac{f_0}{\omega_0^2} = \dfrac{F_0}{m\omega_0^2}$ at low frequencies so that a and θ can be expressed

$$a = a_0\frac{\omega_0/p}{[\{(\omega_0/p) - (p/\omega_0)\}^2 + 1/Q^2]^{\frac{1}{2}}} \text{ and } \tan\theta = \frac{1}{(\omega_0/p) - (p/\omega_0)} \qquad ...(4)$$

Using eq. (4), amplitude (a) and phase lag (θ) versus driving frequency (p) curves are shown in **fig. 7** for different values of Q. The amplitude acquires maximum value (a_m) at the amplitude resonant frequency (p_r). Using eq. (11) and (12) of Art. 8.6, we can express a_m and p_r in terms of Q as follows (when damping is considered):

$$\frac{a_{max}}{a_0} = \frac{Q}{\sqrt{1-(1/4Q^2)}} \text{ and } \frac{p_r}{\omega_0} = \sqrt{1-\frac{1}{2Q^2}} \quad ...(5)$$

For different values of Q, each curve has a maximum value a_m, provided $Q \le 1/\sqrt{2}$ because clearly from eq. (5), then there are no oscillations. We find that when Q is high (or damping is small), the value of a_{max} is high and for small Q (or high damping), it is small. The amplitude resonant frequency p_r is less than ω_0, differing from it more for smaller Q. However we consider, generally, lightly damped or high Q systems, so that $a_{max}/a_0 = Q$ and $p_r/\omega_0 = 1$ nearly.

9. Velocity Resonance

The velocity of the driven oscillator is

$$v = \frac{dx}{dt} = \frac{f_0 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}} \cos(pt - \theta) \quad \text{[from eq. (8), Art. 11.2]}$$

or $$v = v_0 \sin(pt - \theta + \pi/2) \quad ...(1)$$

where $$v_0 = \frac{f_0 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}} = \frac{f_0}{\sqrt{(p - \omega_0^2/p)^2 + 4b^2}}$$ is the *velocity amplitude*

and $$\theta = \tan^{-1}\left(\frac{2bp}{\omega_0^2 - p^2}\right).$$

Thus the velocity amplitude v_0 varies with the driven frequency p **[fig. 8]**. When $p = 0$, $v_0 = 0$ and when $p = \omega_0$, v_0 attains the maximum value. Hence at the frequency $p = \omega_0 = \sqrt{k/m}$ of the impressed force, the velocity has the maximum value and we call it **velocity resonance**. At velocity resonance,

$$\theta = \tan^{-1}\left(\frac{2bp}{0}\right) = \frac{\pi}{2},$$

$\therefore$ Velocity phase constant

$$= -\theta + \pi/2 = -\pi/2 + \pi/2 = 0.$$

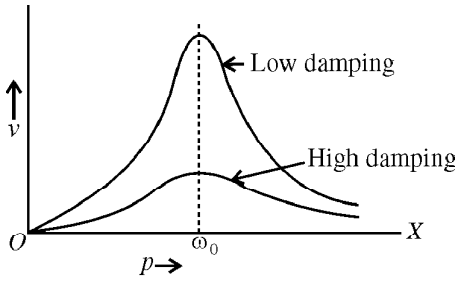


Fig. 8: Variation of velocity amplitude with driven frequency (p).

This means that at velocity resonance, the velocity of the oscillator is in phase with the applied force. This is the most favourable situation for transfer of energy from the applied force to the oscillator, because the rate of work done on the oscillator by the impressed force is Fv, which is always positive for F and v in phase.

When $p \gtrless \omega_0$, the velocity amplitude is smaller than at $p = \omega_0$.

10. Power Absorption and Resonance Width

An oscillating physical system loses energy in doing work against damping forces. In a driven harmonic oscillator, this loss of energy is supplied by the applied harmonic force and hence the oscillations of the system are maintained at constant amplitude. One will be interested in knowing the rate at which the energy must be supplied to maintain the vibrations. The rate of supplying the energy or power absorption by the oscillating system at any instant t is given by

$$P = \frac{dW}{dt} = F\frac{dx}{dt} = Fv \quad ...(1)$$

Now, $$F = mf_0 \sin pt \text{ and } v = \frac{dx}{dt} = \frac{pf_0\cos(pt - \theta)}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}}$$

Hence at any time the energy absorbed by the oscillating system is

$$P = F\frac{dx}{dt} = mf_0 \sin pt \cdot \frac{pf_0 \cos(pt-\theta)}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}} \quad ...(2)$$

Hence the average power absorbed is

$$P_{av} = \left(F\frac{dx}{dt}\right)_{av} = \frac{mf_0^2 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}} [\sin pt \cos(pt-\theta)]_{av} \quad ...(3)$$

Now the average of sin *pt* cos (*pt*-θ) for the period *T*

$$= \frac{1}{T}\int_0^T \sin pt \cos(pt-\theta)\, dt = \frac{1}{T}\int_0^T \frac{1}{2}\{\sin(2pt-\theta) + \sin\theta\} dt = \frac{\sin\theta}{2}$$

$$= \frac{1}{2}\frac{2bp}{\sqrt{(\omega_0^2 - p^2)^2 + 4b^2p^2}} \quad \text{[from eq. (7); Art 8.6]}$$

$$\therefore \quad P_{av} = \frac{1}{2}mf_0^2 \frac{2bp^2}{(\omega_0^2 - p^2)^2 + 4b^2p^2} \quad ...(4)$$

Replacing $\frac{f_0^2 p^2}{(\omega_0^2 - p^2)^2 + 4b^2p^2}$ by v_0^2, we have

$$P_{av} = mbv_0^2 = \frac{mv_0^2}{2\tau} \text{ or } \frac{\gamma v_0^2}{2} \quad ...(5)$$

We see from eq. (4) that when $p = \omega_0$, the absorbed average power is maximum, say P_m, given by

$$P_m = \frac{1}{4}\frac{mf_0^2}{b} = \frac{1}{2}mf_0^2\tau. \quad ...(6)$$

Thus *the power absorbed is maximum at the frequency of velocity resonance and not at the frequency of amplitude resonance*. A graph between P_{av} versus *p* for *a* particular value of damping constant *b* (or $Q = \omega_0\tau = \omega_0/2b$) is shown in **fig. 9**.

When steady state is attained, the average power dissipated* should be the same as the average power supplied by the driving force.

Resonance width: If p_h is the value of the driving frequency when the average power falls to half the value at resonance, the *half-width of the power resonance* curve is given by

$$\Delta p = |p_h - \omega_0| \quad ...(7)$$

The power drops to its half the maximum value, when

$$\frac{1}{2}P_m = (P_{av})_{p_h}$$

$$\frac{1}{2}\frac{mf_0^2}{4b} = \frac{1}{2}mf_0^2 \frac{2bp_h^2}{(\omega_0^2 - p_h^2)^2 + 4b^2p_h^2}$$

or $$8b^2p_h^2 = (\omega_0^2 - p_h^2)^2 + 4b^2p_h^2$$

or $$\omega_0^2 - p_h^2 = \pm 2bp_h$$

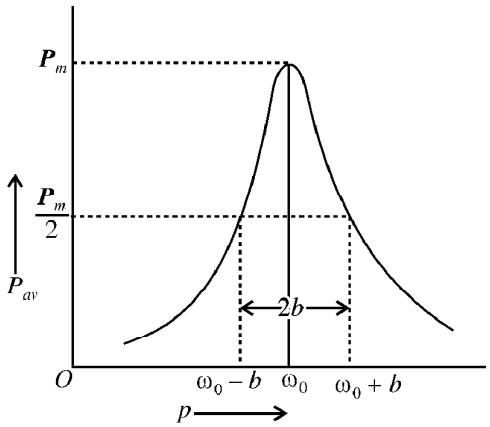


Fig. 9: Average power absorbed (p_{av}) as a function of driving frequency: Full width of the resonance curve $2\Delta p = 2b$ or $1/\tau$; $Q = \omega_0 t$.

---

* Instantaneous power dissipated = (force) $dx/dt = 2bm(dx/dt)^2 = 2bm \cdot \frac{f_0^2 p^2}{(\omega_0^2 - p_0^2)^2 + 4b^2p^2}\cos^2(pt-\theta)$

Average power dissipated = $\frac{1}{2}mf_0^2 \frac{2bp^2}{(\omega_0^2 - p^2)^2 + 4b^2p^2}$. $\quad [\because \{\cos^2(pt-\theta)_{av}\} = \frac{1}{2}]$

or $\omega_0 - p_h = \pm \frac{2bp_h}{\omega_0 + p_h} = \pm \frac{2b}{1+\omega_0/p_h} \approx \pm b$ or $\frac{1}{2\tau}$ [taking $\omega_0/p_h = 1$ approx.]

$\therefore$ **Half width** of the power versus frequency curve

$$\Delta p = |\omega_0 - p_h| = \frac{1}{2\tau} = \frac{\omega_0}{2Q} \quad [\because Q = \omega_0\tau] \quad ...(8)$$

The quantity $2\Delta p$ is called the **band-width** or **full-width** at the half maximum power and is given by

$$2\Delta p = \frac{\omega_0}{Q} = \frac{1}{\tau} \text{ or } 2b \quad ...(9)$$

Thus the resonance full-width for forced oscillations is equal to the inverse of decay time (τ) for free oscillations.

Also, $$Q = \omega_0\tau = \frac{\omega_0}{2\Delta p} = \frac{\text{Frequency at resonance}}{\text{Full width at half maximum power}} \quad ...(10)$$

Alternatively, $$Q = 2\pi \frac{\text{Average energy stored}}{\text{Average energy dissipated per cycle}} \quad ...(11)$$

At any instant, stored energy of oscillator = K.E + P.E.

Time averaged stored energy, $E = \langle \text{K.E.} \rangle + \langle \text{P.E.} \rangle = \frac{1}{2} < m\left(\frac{dx}{dt}\right)^2 > + < \frac{1}{2}kx^2 >$

$$= \frac{1}{2}mp^2a^2 < \cos^2(pt-\theta) > + \frac{1}{2}m\omega_0^2a^2 < \sin^2(pt-\theta) >$$

$$= \frac{1}{4}mp^2a^2 + \frac{1}{4}m\omega_0^2a^2 = \frac{1}{4}m(p^2+\omega_0^2)a^2 \quad ...(12)$$

At $p \approx \omega_0$, $$E = \frac{1}{2}m\omega_0^2a^2$$

Average energy dissipated per sec. $= < \gamma \frac{dx}{dt} > = \gamma\, p^2a^2 < \cos^2(pt-\theta) > = \frac{m}{2\tau} p^2a^2 \quad ...(13)$

Average energy dissipated per cycle (at $p \approx \omega_0$) $= \frac{m\omega_0^2a^2}{2\tau} \cdot \frac{2\pi}{\omega_0} \quad ...(14)$

$\therefore$ $$Q = 2\pi \frac{\frac{1}{2}m\omega_0^2a^2}{\frac{m\omega_0^2a^2}{2\tau} \cdot \frac{2\pi}{\omega_0}} = \omega_0\tau = \frac{\omega_0}{2\Delta p} \quad ...(15)$$

Sharpness of resonance and quality factor: We may use the term, sharpness of resonance, to refer the rapid rate of fall in power with the frequency on each side of the resonant frequency (ω_0). Thus for small band-width ($2\Delta p$) or high Q system, the resonance, is said to be **sharp**. For low Q system, the band-width will be more and the power falls off slowly. The resonance in such a case is said to be **flat**. Thus Q is a measure of the sharpness of the resonance.

11. Driven LCR Circuit

Driven LCR circuit is an example of the driven harmonic oscillator in which the oscillations are sustained by an alternating electromotive force. Consider a circuit containing the inductance L, resistance R and capacity C in series. An alternating E.M.F. $E = E_0 \sin pt$ is applied to this circuit, where E_0 is the peak value **[fig. 10]**. If at any instant t, q is the charge on the condenser C, then the potential difference across its plates is q/C. If the current in the circuit at this instant is i, then the self-induced

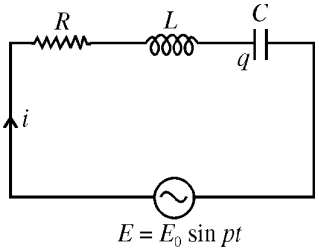


Fig. 10.

E.M.F. in the inductance is $-L\,di/dt$. According to Ohm's law, the potential difference between the ends of the resistance is RI. Hence

$$\text{Total potential difference} = Ri + \frac{q}{C} = \text{total E.M.F.} = E - L di/dt, \text{ i.e. } L\frac{di}{dt} + Ri + \frac{q}{C} = E \quad ...(1)$$

or
$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E_0}{L}\sin pt. \qquad \left[\because i = \frac{dq}{dt}\right] \quad ...(2)$$

This equation is similar to the equation of driven harmonic oscillator having R/L in place of $2b$, $1/LC$ for ω_0^2 and E_0/L for f_0. Hence the steady state solution of the above equation is

$$q = \frac{E_0/L}{\sqrt{\left(\frac{1}{LC} - p^2\right)^2 + \left(\frac{pR}{L}\right)^2}}\sin(pt - \theta) \text{ with } \theta = \tan^{-1}\frac{pR/L}{\frac{1}{LC} - p^2} \quad ...(3)$$

Differentiating q in eq. (3) with respect to t, we can find the current $i = dq/dt$ in the circuit. Practically, we require the knowledge of the current in the circuit*.

$$i = \frac{E_0 p/L}{\sqrt{\left(\frac{1}{LC} - pL\right)^2 + R^2}}\cos(pt - \theta) \text{ or } i = \frac{E_0}{\sqrt{R^2 + \left(pL - \frac{1}{pC}\right)^2}}\sin(pt - \phi) \quad ...(4a)$$

where
$$\phi = \theta - \frac{\pi}{2} = \tan^{-1}\frac{pL - 1/pC}{R} \quad \left[\because \tan\phi = -\cot\theta = \frac{p^2 - 1/LC}{pR/L} = \frac{pL - 1/pC}{R}\right] ...(4b)$$

which represents that the current i lags is phase ϕ from the E.M.F. E.

The quantity $\sqrt{R^2 + (pL - 1/pC)^2}$ is called the **impedance** of the circuit and is represented by Z. Its unit is Ohm. Hence

$$i = \frac{E_0}{Z}\sin(pt - \phi) \text{ or } i = i_0 \sin(pt - \phi) \quad ...(5)$$

where $i_0 = E_0/Z$ is the peak value of the current.

The quantity $(pL - 1/pC)$ is called the **reactance** of the circuit and is denoted by X. Therefore

$$\text{Impedance, } Z = \sqrt{R^2 + X^2} = \sqrt{(\text{Resistance})^2 + (\text{Reactance})^2} \quad ...(6)$$

Now, the current amplitude $i_0 = \frac{E_0}{Z} = \frac{E_0}{\sqrt{R^2 + X^2}}$ is smaller for the larger value of Z. The expression for
$$Z = \sqrt{R^2 + (pL - 1/pC)^2} \quad ...(7)$$
depends upon the frequency (p) of the applied alternating E.M.F. We will consider below the important cases, when the driven frequency is increased. We denote $\omega_0 = 1/\sqrt{LC}$ as the undamped frequency of the circuit.

Case I: $p << \omega_0$, the term $-1/pC$ in impedance is **dominant** and is very large and $Z = \sqrt{R^2 + (pL - 1/pC)^2} \approx 1/pC$.

Therefore, the current amplitude is (when $p \to 0$)

---

* **A shorter procedure for solution of eq.** (1) : Let the solution be $i = i_0 \sin(pt - \phi)$. Hence $q = \int i dt = \frac{i_0}{p}\cos(pt - \phi)$
Substituting the values of di/dt, i and q in eq. (1) and proceeding as before, the values of i and ϕ can be calculated.

$$i_0 = \frac{E_0}{1/pC} \text{ and } \phi = -\frac{\pi}{2} \quad ...(8)$$

Thus, for low values of p, the current amplitude has a small value. On increasing the driven frequency (p) gradually, the magnitude of reactance ($pL - 1/pC$) and hence the impedance Z decreases resulting in the increase of the current amplitude.

Case II: When $p = \omega_0 = 1/\sqrt{LC}$, the reactance

$$X = pL - \frac{1}{pC} = \frac{L}{\sqrt{LC}} - \frac{\sqrt{LC}}{C} = \sqrt{\frac{L}{C}} - \sqrt{\frac{L}{C}} = 0 \quad ...(9)$$

and so the impedance Z is minimum and is equal to R. In consequence the current amplitude $i_0 = E_0/Z = E_0/R$ becomes maximum. This is the case of **electrical resonance**. If $R = 0$, then the current amplitude at resonance becomes infinite. The **resonant frequency** of the circuit is given by

$$\nu_r = \frac{p}{2\pi} = \frac{F_0}{m} = \frac{1}{2\pi\sqrt{LC}} \quad ...(10)$$

In such a case, $\phi = 0$ and $i = \frac{E_0}{R} \sin pt$ ...(11)

Thus at *resonance the current and the applied E.M.F. are in the same phase.*

Case III: When $p >> \omega_0$, the term $L\omega$ in the expression for Z becomes dominant and very large and then.

$$i_0 = E_0/pL \text{ and } \phi = \pi/2 \quad ...(12)$$

The variation of i_0 with p is shown in **fig. 11** for different values of R. The smaller value or R, the sharper is the resonance.The variation of phase difference ϕ between the current and the E.M.F. is shown in **fig. 12** The current lags in phase for $p < \omega_0$ and exceeds in phase for $p > \omega_0$.

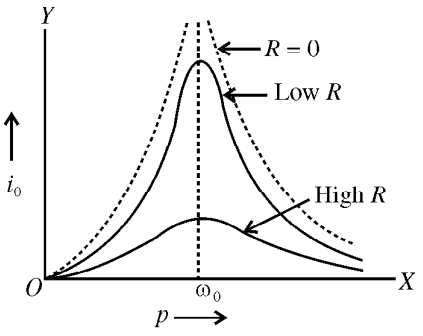


Fig. 11.

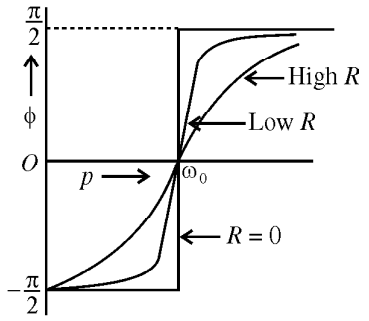


Fig. 12.

Power in A.C. circuits: The power of an electrical circuit is defined as the rate of its doing work and at any time it is measured by the product of the current and the potential difference. If at any instant i is the current and E, the E.M.F., then the instantaneous power delivered by the source to the circuit is

$$P = Ei. \quad ...(13)$$

Now, let us consider the power relations by taking LCR circuit. The instantaneous power delivered by the source to the circuit

$$P = Ei = E_0 \sin pt \; i_0 \sin (pt - \phi) = \frac{E_0 i_0}{2} [\cos \phi - \cos (2pt - \phi)] \quad ...(14)$$

where, the average value of $\cos (2pt - \phi)$ for one cycle is zero, hence the average power of an A.C. circuit is given by

$$P_{av} = \frac{E_0 i_0}{2} \cos \phi = \frac{E_0}{\sqrt{2}} \frac{i_0}{\sqrt{2}} \cos \phi = E_{rms} \, i_{rms} \cos \phi \quad ...(15)$$

where, $E_{rms} = \frac{E_0}{\sqrt{2}}$ and $i_{rms} = \frac{i_0}{\sqrt{2}}$.

This cos ϕ in eq. (15) is called the **power factor** of the circuit and it will be given by

$$\cos\ \phi\ =\ \frac{R}{\sqrt{R+\left(pL-1/pC\right)^2}}=\ \frac{R}{Z} \qquad \text{[From eq. (4b)]}$$

At resonance, $\phi = 0$ so the average power delivered to the circuit is maximum and then

$$(P_{av})_{max}\ =(P_{av})_{res}=\ \tfrac{1}{2}E_0 i_0=\ \tfrac{1}{2}\frac{E_0^2}{R}=\tfrac{1}{2}i_0^2 R \qquad ...(16)$$

But, $$i_{rms}\ =\frac{i_0}{\sqrt{2}}=\frac{1}{\sqrt{2}}\cdot\frac{E_0}{Z}=\frac{E_{rms}}{Z}$$

$\therefore$ Power in A.C. circuit $= (i_{rms}\ Z).\ i_{rms}\ R/Z = i_{rms}{}^2\ R$ ....(17)

Thus in an A.C. circuit, the power appears only in the form of heat.

Analogous to mechanical oscillator, we can show easily that

$$Q\ =\ \omega_0\tau\ =\frac{L\omega_0}{R}\ =\ \frac{\omega_0}{2\Delta p}\ =\ \frac{\text{Frequency at resonance}}{\text{Full width at half maximum power}}. \qquad ...(18)$$

***Q*-Factor:** Similar to mechanical oscillator, the quality factor is given by

$$Q\ =\ \omega_0\tau=\ \frac{\omega_0 L}{R},\ \text{where}\ \omega_0=\ \frac{1}{\sqrt{LC}} \qquad ...(19)$$

Full width or the band-width of the power resonance curve is given by

$$2\Delta p\ =\ \frac{1}{\tau}=\frac{R}{L}=\frac{\omega_0}{Q} \qquad ...(20)$$

Thus $$Q\ =\ \frac{\omega_0}{2\Delta\omega}\ =\ \frac{\text{Frequency at resonance}}{\text{Full width at half maximum power}} \qquad ...(21)$$

The full width $2\Delta\omega$ is small for high Q. Circuits with Q of the order of 10^2 to 10^4 are quite common. The resonant circuit in a radio receiver with Q of the order of 10^2 to 10^3 may select a particular station and discard the others.

12. Mechanical and Electrical Analogues

The differential equation for forced oscillations if a mechanical oscillator is

$$m\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+kx\ =\ F_0\ \sin\ pt \qquad ...(1)$$

The differential equation for a *LCR* circuit, driven by an alternating electromotive force is

$$L\frac{a^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}\ =\ E_0\ \sin\ pt \qquad ...(2)$$

Equation (1) and (2) are similar and hence allow analogies between the mechanical and electrical quantities. A look at the equations shows that the mass is analogous to the self-inductance L, the frictional force per unit velocity or mechanical resistance γ to the electrical resistance R, the force constant to the inverse of capacitance C, the force F ($= F_0 \sin pt$) to the voltage E ($= E_0 \sin pt$), the displacement x to the charge q and the velocity dx/dt to the current dq/dt.

The velocity (dx/dt) of force oscillator in the steady state is given by

$$v\ =\ \frac{f_0}{\sqrt{(p-\omega_0{}^2/p)^2+4b^2}}\cos(pt-\theta)$$

where $$f_0\ =\ \frac{F_0}{m},\ \omega_0{}^2=\ \frac{k}{m}\ \text{and}\ b\ =\ \frac{\gamma}{2m}$$

Hence $$v = \frac{F_0/m}{\sqrt{(p - k/mp)^2 + (\gamma/m)^2}} \cos(pt - \theta)$$

or $$v = \frac{F_0}{\sqrt{(pm - k/p)^2 + \gamma^2}} \cos(pt - \theta) \quad ...(3)$$

The current (dq/dt) in a driven *LCR* circuit is given by

$$i = \frac{E_0}{\sqrt{(pL - 1/pC)^2 + R^2}} \cos(pt - \theta) \quad ...(4)$$

In an *AC* electrical circuit the quantity $\sqrt{(pL - 1/pC)^2 + R^2}$ is called impedance and is denoted by Z i.e.,

$$Z = \sqrt{(pL - 1/pC)^2 + R^2} \quad ...(5)$$

Equation (3) and (4) are similar, hence the impedance of mechanical oscillator is denoted by Z_m, given by

$$Z_m = \sqrt{(pm - k/p)^2 + \gamma^2} \quad ...(6)$$

The mechanical impedance is often expressed in mechanical ohm similar to ohm for electrical impedance.

For ready reference, mechanical quantities and the corresponding electrical analogues are given in the following table.

Table: Analogous Mechanical and Electrical Quantities

Mechanical Quantities	Electrical Quantities
1. Mass (m)	1. Inductance (L)
2. Force (F)	2. Voltage (E)
3. Mechanical resistance (γ)	3. Electrical resistance (R)
4. Displacement (x)	4. Charge (q)
5. Velocity ($v = dx/dt$)	5. Current ($i = dq/dt$)
6. Compliance ($1/k$)*	6. Capacitance (C)
7. Mechanical impedance (Z_m)	7. Electrical impedance (Z)

13. Free, Damped and Forced Vibrations – A Comparision

Free Vibrations (Undamped harmonic oscillator)	Damped Free Vibrations (Damped harmonic oscillator)	Forced Vibrations (Driven harmonic oscillator)
1. The vibrations of a system free from external force are called free vibrations.	1. The free vibrations of a sysem in the presence of frictional forces are called damped free vibrations.	1. The vibrations of a system under the action of an external periodic force are called forced vibrations.
2. In case of free vibrations, the energy of vibrating system remains constant.	2. In case of damped free vibrations, the energy of the vibrating system decreases with time because of doing work against damping forces.	2. In case of forced vibrations, the applied harmonic force supplies the energy to the oscillator periodically to compensate for the loss of energy due to damping forces.
3. Such vibrations have constant amplitude and continue indefinitely.	3. The amplitude of such vibrations decreases with time and ultimately the vibrating system comes to rest.	3. Such vibrations with constant amplitude continue as long as the external periodic force acts.

4. The different equation for undamped harmonic oscillator is $\frac{d^2x}{dt^2} + \omega_0{}^2 x = 0$ where $\omega_0 = \sqrt{k/m}$ 5. Period $T = \frac{2\pi}{\omega_0}$	4. The different equation for damped harmonic oscillator is $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2{}_0 x = 0$ where b is damping constant 5. Period $T = \frac{2\pi}{\sqrt{\omega^2{}_0 - b^2}}$	4. The differential equation for forced oscillator is $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2{}_0 x = f_0 \sin pt$ where p is angular frequency of the external periodic force. 5. The system vibrates with the frequency p of the applied harmonic force in the steady state.

14. Transient State and Steady State

In this chapter we have so far studied the steady state behaviour of an oscillating system, acted by a harmonic driving force. In fact, in a practical situation, the driving force is first switched on at some instant on a damped harmonic oscillator. The frequency (p) of driving force in general different to the natural frequency (ω) of the damped harmonic oscillator. In the beginning, the oscillator tries to vibrate with its natural frequency (ω) and the applied (driven) periodic force tends to impose its own frequency (p) upon it. Hence for some time, the oscillating system is in the **transient state** and the free oscillations of frequency ω die out and ultimately the system starts oscillating with a constant amplitude with the frequency (p) of the driven harmonic force. This is called the **steady state.** In the transient state, the oscillations of the system take place under the superposition of two oscillations of frequenclies $\omega = \sqrt{\omega_0^2 - b^2}$ and p.

We consider the system to be at rest with no initial displacement at $t = 0$, the time when the driving force is switched on. We are interested in knowing the nature of the transient motion just after the instant $t = 0$.

Mathematically, the problem is to find the general solution of the differential equation of the driven harmonic oscillator:

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt \qquad [f_0 = F_0/m] \qquad ...(1)$$

The steady state solution is not the general solution. According to the therory of differential equations, the general solution $x(t)$ of eq. (1) is the superposition of two solutions $x_1(t)$ and $x_2(t)$, given by

(1) $x_1(t) = a \sin(\omega t - \theta)$, which is the steady state solution (particular solution) of eq. (1).

(2) $x_2(t) = a_0 e^{-bt} \sin(\omega t + \phi)$, which is the general solution of the differential equation,

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0.$$

Thus the general solution of eq. (1) is

$$x = x_1 + x_2 \text{ or } x = a \sin(pt - \theta) + a_0 e^{-bt} \sin(\omega t + \phi) \qquad ...(2)$$

We impose the initial conditions that at $t = 0$, $x = 0$ and $dx/dt = 0$ on the general solution x, given by eq. (2) and its time derivative dx/dt. Now,

$$\frac{dx}{dt} = pa \cos(pt - \theta) + a_0 e^{-bt} [-b \sin(\omega t + \phi) + \omega \cos(\omega t + \phi)] \qquad ...(3)$$

Imposing the, initial conditions on eqs. (2) and (3), we obtain

$$0 = -a \sin\theta + a_0 \sin\phi \text{ and } 0 = pa \cos\theta - a_0 b \sin\phi + a_0 \omega \cos\phi \qquad ...(4)$$

We are interested here in the case in which the *damping is weak and the driving fequency is close to the resonant frequency* i.e., $b << \omega$ and $p \approx \omega$. In such a case, the conditions (4) become

$$a_0 \sin \phi = a \sin \theta \text{ and } a_0 \cos \phi = -a \cos \theta \quad ...(5)$$

Dividing $\tan \phi = -\tan \theta$ or $\phi = \pi - \theta$. So that $a_0 = a$ ...(6)

Hence the general solution is

$$x = a \sin (pt - \theta) - a_0 e^{-bt} \sin (\omega t - \theta) \quad ...(7)$$

Now, first let us consider the case of zero damping ($b = 0$) or ideal oscillator, then from eq. (9) [Art. 8.6],

$$a = \frac{F_0}{m(\omega_0^2 - p^2)}, \ \theta = 0 \text{ and } \omega = \omega_0 \quad ...(8)$$

Hence eq. (7) takes the form

$$x = \frac{F_0}{m(\omega_0^2 - p^2)} (\sin pt - \sin \omega_0 t) \text{ or } x = \frac{2F_0}{m(\omega_0^2 - p^2)} \cos \frac{p - \omega_0}{2} t \sin \frac{p + \omega_0}{2} t \quad ...(9)$$

or $$x = a_m(t) \sin \omega_a t \quad ...(10)$$

where $$a_m(t) = \frac{2F_0}{m(\omega_0^2 - p^2)} \cos \omega_m t \text{ with } \omega_m = \frac{p - \omega_0}{2} \text{ and } \omega_a = \frac{p + \omega_0}{2}$$

This is the case when we were studying the superposition of two simple harmonic motions of two slightly different frequencies and phenomenon of modulation and beats.

Thus the beats occur between the natural and driven oscillations at a frequency $\omega_0 - p$ and in case of zero damping, phenomenon of beat goes on continuously.

Next, we consider the case, when the damping is present i.e., the case of damped harmonic oscillator alongwith the applied harmonic force we see from eq. (7) that at $t = 0$, the displacement is a suerposition of two harmonic terms with the driving frequency (p) close to the natural frequency (ω). Thus the initial

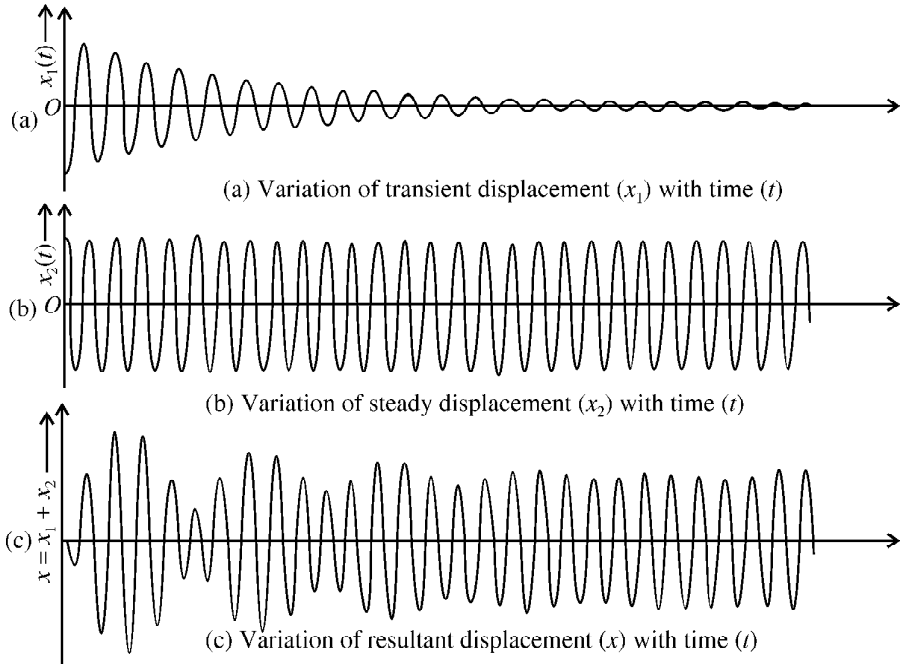


Fig. 13: Variation of displacement of a driven escillator (for low damping) with time from starting at $t = 0$.

motion consists of beats between the free and driven oscillations. If there were no damping, these beats would continue forever, as we have seen above. However due to damping the second term of eq. (7)

dimnishes with time, resulting in weak beats. These are called transient beats and the oscillator is said to be in the **transient state**. After a sufficiently long time compared to the decay time τ, the **steady state solution** becomes dominant [i.e. first term of eq. (7)] and the oscillator executes vibrations of constant amplitude (with no beats) with the frequency of driven harmonic force **[fig. 13]**. In the beginning, due to damping, the oscillator adjusts gradually its phase relative to the driving harmonic force and the relative phase is stabilized in the steady state.

Ex. 1. *The amplitude of forced vibrations is given by*

$$a = \frac{f}{\sqrt{(\omega_0^2 - p^2)^2 + p^2/\tau^2}}.$$

If $Q = 40$, *calculate the value of* A/A_{max} *for* $p/\omega_0 = 0.99$.

Sol. $Q = \omega_0\tau = 50$ or $\tau = 50/\omega_0$

and $$a = \frac{f}{\sqrt{(\omega_0^2 - p^2)^2 + p^2/\tau^2}} = \frac{f}{\omega_0^2\sqrt{\left(1-\frac{p^2}{\omega_0^2}\right) + \frac{1}{2500}\frac{p^2}{\omega_0^2}}}$$

For small damping, at resonance $a_{max} = \frac{f}{\omega_0^2 \times 1/50} = \frac{50f}{\omega_0^2}$

When $p/\omega_0 = 0.99$, $\frac{a}{a_{max}} = \frac{1}{50}\frac{1}{\sqrt{(1-0.99)^2 + \frac{1}{2500}(0.99)^2}} =$ **0.71**.

Ex. 2. *In case of a forced harmonic oscillator, the amplitude of vibrations increases from* 0.02 *mm at very low frequencies to a value* 5 *mm at the frequency* 100 *Hz. Find* (i) *Q-factor of the system,* (ii) *damping constant b and relaxation time and* (iii) *half width of amplitude resonance curve.*

Sol. (i) $Q = \omega_0\tau = \frac{\omega_0}{2b} = \frac{a_{max}}{a \text{ at zero or very low frequency}} = \frac{5}{0.02} =$ **250**.

(ii) Resonant frequency = 100 Hz, $\therefore\ \omega_0 = 2\pi \times 100$

Hence $\tau = \frac{Q}{\omega_0} = \frac{250}{2\pi\times100} =$ **0.4 sec.** and $b = \frac{1}{2\tau} = \frac{1}{2\times0.4} =$ **1.25 per sec**.

(iii) Half width $\Delta p = \sqrt{3}\times1.25 =$ **2.2 rad/sec**.

Ex. 3. *If a constant torque of* 8×10^{-5} *N-m produces an angular displacement of* 2^o *in a torsion pendulum whose moment of inertia is* 10^{-3} *kg-m*^{2}, *calculate the frequency of the periodic external torque which will produce resonance.*

Sol. Angular displacement = $2^o = 2 \times \pi/180$ radian; applied couple = 8×10^{-5} N-m

Restoring couple for unit displacement = $\frac{8\times10^{-5}}{2\pi} \times 180$ N-m/rad.

At resonance, frequency of the torque = approx. natural frequency

$$= \frac{1}{2\pi}\sqrt{\frac{C}{I}} = \frac{1}{2\pi}\sqrt{\frac{8\times10^{-5}\times180}{2\pi\times10^{-3}}} =$$ **0.24 Hz**.

Ex. 4. *In case of a driven harmonic oscillator, the amplitude of vibrations increases from* 0.1 *mm at very low frequencies to a maximum value of* 5 *mm at the frequency* 200 *Hz. Find Q-factor, damping constant b and relaxation time.*

Sol. $Q = \dfrac{\text{Amplitude at resonance}}{\text{Amplitude at zero or very low frequency}} = \dfrac{5}{0.1} = \mathbf{50}$.

Here, resonant frequency $\nu_0 = 200$ Hz, $\therefore\ \omega_0 = 2\pi \times 200$

Therefore, $\tau = \dfrac{Q}{\omega_0} = \dfrac{50}{2\pi \times 200} = \mathbf{0.04\ sec.}$ and $b = \dfrac{1}{2\tau} = \mathbf{12.6\ per\ sec}$.

Ex. 5. *Find the average energy stored in a* 0.1 *kg mass, attached to a spring. The mass is oscillating with an amplitude of* 5 *cm in resonance with a harmonic force of frequency* 30 *Hz. If the quality factor of the oscillator is* 100, *calculate the power losses.*

Sol. At resonance, $p = \omega_0$, the average energy stored

$$= \frac{1}{2} m \omega_0^2 a^2 = \frac{1}{2} \times 0.1\ (2\pi \times 30)^2 \times (5 \times 10^{-2})^2 = \mathbf{4.44\ J}.$$

Now, $Q = \dfrac{\text{Average energy stored}}{\text{Average energy dissipated per period}}$

Period, T = 1/30 sec; $\therefore$ Average energy dissipated in 1/30 sec. $= \dfrac{2\pi \times 4.44}{100} = \mathbf{0.279\ J}$.

Average energy dissipated per sec. or power losses $= 30 \times 0.279 = \mathbf{8.37\ J.}$

Ex. 6. *In case of a forced harmonic oscillator of mass* 10 gm *the amplitude of vibration is* 0.2 mm *at very low frequencies. As the frequency is varied, the amplitude value increases to a maximum value of* 20 mm *for a frequency of* 200 Hz *and thereafter decreases with further increase in frequency. Find (i) the Q factor of the system (ii) the damping constant and (iii) the half width of the resonance curve.*

Sol. (i) $Q = \dfrac{\text{Maximum amplitude at resonance}}{\text{Amplitude at very low frequencies}} = \dfrac{20 \times 10^{-3}}{0.2 \times 10^{-3}} = \mathbf{100}$

(ii) Resonant angular frequency $\omega_0 = 2\pi \times 200 = 400\pi\ \text{sec}^{-1}$

Now, $Q = \omega_0 \tau = \dfrac{\omega_0}{2b}$

$\therefore$ Damping constant $b = \dfrac{\omega_0}{2Q} = \dfrac{400\pi}{2 \times 100} = \dfrac{400 \times 3.14}{2 \times 100} = \mathbf{6.28\ sec^{-1}}$

(iii) Half band width $\Delta p = \sqrt{3}\ b = \sqrt{3} \times 6.28 = 1.732 \times 6.28 = \mathbf{10.88\ rad/sec.}$

Ex. 7. *An A. C. supply of frequency* 10000 *and* 110 *volts is connected across a circuit containing a resistance of* 10 *ohms, an inductance of* 10^{-3} *henry and a capacity of* 1 μF. *Find the value of the current. What must be the value of the capacity in order that the current may be maximum.*

Sol. Impedance, $Z = \sqrt{R^2 + (pL - 1/pC)^2} = \sqrt{R^2 + (2\pi\nu L - 1/2\pi\nu C)^2}$

$$= \sqrt{10^2 + \left(2\pi \times 10000 \times 10^{-3} - \frac{1}{2\pi \times 10000 \times 10^{-6}}\right)} = 48 \text{ ohms approx.}$$

$\therefore$ Current = e.m.f./Z = 110/48 = **2.293 amp**.

Maximum current will flow when the total resistance is zero and resonance is produced. Hence the maximum current will be when $pL - 1/pC = 0$ i.e., when

$$C = \frac{1}{Lp^2} = \frac{1}{L \times 4\pi^2 \nu^2} = \frac{1}{10^{-3} \times 4 \times \pi^2 \times (10^4)^2} = \mathbf{0.253 \times 10^{-6}\ farad}.$$

Ex. 8. *An A.C. potential of* 1×10^5 *sec*$^{-1}$ *frequency and* 1.0 *volt amplitude is applied to a series LCR circuit. If* R = 2.0 *ohm,* L = 0.50 *milihenry, calculate the value of C to secure resonance. Also calculate the rms value of current and peak potenial difference across the condenser.*

Sol. Capacity, required to produce resonance, is given by

$$C = \frac{1}{4\pi^2 n^2 L} = \frac{1}{4\times(3.14)^2\times(10^5)^2\times 0.5\times 10^{-3}} = \mathbf{5 \times 10^{-9}\ farad.}$$

$$I_{rms} = \frac{E_{rms}}{R} = \frac{E_0}{R\sqrt{2}} = \mathbf{\frac{1}{2\sqrt{2}}\ ampere} \qquad \text{[at resonance } Z = R\text{]}$$

Peak potential differential across the condenser = peak current × reactance across condenser

$$= \frac{E_0}{R}\times\frac{1}{pC} = \frac{1}{2}\times\frac{1}{10^5\times 5\times 10^{-9}} = \mathbf{10^3\ volts}.$$

QUESTIONS

Long Answer Questions

1. What are damped oscillations? Solve the differential equation of a damped harmonic oscillator. Discuss the cases when it is over-damped and critically damped. **(A.N.U. 2007)**

2. Discuss the differential equation of damped oscillator and obtain the solution under different conditions. **(A.N.U. 2014)**

3. What are damped oscillations? Obtain the differential equation and discuss its solution under different conditions. **(S.U. 2014; A.N.U. 2013)**

4. (a) Derive the equation of motion of damped harmonic oscillator and obtain its solution.
(b) Discuss the conditions under which the oscillations are under-damped, over-damped and critically-damped. **(O.U. March 2007)**

5. For the following cases represent the variation of displacement against time graphically:
(1) Undamped harmonic oscillator
(2) Overdamped simple harmonic oscillator
(3) Critically damped simple harmonic oscillator.
Mention the corresponding solutions alongside the curves. **(O.U. 2002)**

6. Derive the equation of damped oscillator. Obtain the solution and discuss underdamped condition. **(A.U. 2013)**

7. Solve differential equation of damped harmonic oscillator and explain the motion in differents damping conditions. **(K.U. 2013)**

8. Find the equation of motion for a damped harmonic oscillator and find its solution. Give two examples of damped harmonic oscillations. **(A.N.U. 2006)**

9. Form the differential equation for the damped harmonic motion of an oscillator and obtain its solution. Discuss the conditions under which the oscillations are under-damped, over damped and critically damped. **(Dr. B.R.A.U. 2014; K.U. 2005)**

10. (a) What is a damped harmonic oscillator? Represent graphically the solution of a damped harmonic oscillator.
(b) Explain the terms Logarithmic decrement, Relaxation time and the *Q* factor of a damped harmonic oscillator. **(O.U. 2003)**

11. Explain the terms 'damping' and '*Q*-factor'. Compare a forced mechanical oscillator with an electrical oscillator. **(S.V.U. 2002)**

12. What do you understand by the relaxation time, damping constant and quality factor of a damped harmonic oscillator? Obtain their expressions.

13. Obtain the expressions of average total energy and average power loss for a damped harmonic oscillator.

14. Define damped harmonic motion and damped harmonic oscillator. Discuss energy consideration in damped harmonic motion. **(R.U. 2014)**

15. If the damping constant $b << \omega_0$, prove that

(a) the average total energy $E = \frac{1}{2} m\omega^2 a_0^2 e^{-2bt.}$

(b) average power dissipation $P = E/\tau$.

(c) average rate of doing work by the damping force $-\gamma\, dx/dt$ is equal to the average power dissipation (i.e. to E/τ).

16. A particle is oscillating under a damping force. Show that the average power loss (P) is given by $P = E/\tau$, where E is the average energy and τ is the relaxation time. What happens to the dissipated energy?

17. The amplitude A of forced vibrations in a system is given by

$$A = \frac{F/m}{[(\omega_0^2 - \omega^2)^2 + (\omega/\tau)^2]^{1/2}}.$$ (Proof not required).

Symbols have their usual meanings. Show that in the limit $\omega << \omega_0$ the response is independent of the mass, while in the limit $\omega >> \omega_0$ the response is independent of the spring constant of the system.

18. Set up the differential equation for forced vibrations and explain the three cases:
(1) $p <<$ ω, (ii) $p = \omega$ and $p > > \omega$ where p is angular frequency of the force. **(O.U. 2005)**

19. What are force vibrations? Obtain an expression for the amplitude and phase of a forced oscillator. **(O.U. 2014)**

20. What are forced oscillations? Obtain an expression for the amplitude of forced vibrations and deduce the condition for resonance. **(A.U. 2014)**

21. Give the theory of dirven harmonic oscillator. Show that at resonance the displacement always lags behind the driving force in phase by $\pi/2$, while the velocity at resonance is always in phase with the driving force.

22. What are free, damped and forced oscillators? Obtain the differential equation for the forced oscillator and solve it.

23. What are forced oscillations? Derive the equation of motion of a driven harmonic oscillator and find its solution. **(O.U. 2009)**

24. Derive equation of motion of a force oscillator and solution of the differential equation of forced oscillator. **(Dr. B.R.A.U. 2013)**

25. What are forced vibrations? Set up the equation of motion for a forced oscillator and its solution. Discuss its frequency response. **(K.U. 2012)**

26. If the frequency of the impressed force on an oscillator is changed gradually, then what changes will take place in the response (amplitude)? Draw a graph, representing this variation. Prove that the displacement always lags in phase from the driven force. Show by means of a graph the relation between the frequency of the force and this phase difference.

27. What are forced oscillations? Obtain an expression for the amplitude of forced vibrations and deduce the condition for resonance. **(A.U. March 2007, K.U. 2007)**

28. (a) Derive differential equation of a driven harmonic oscillator.
(b) Distinguish between velocity resonance and amplitude resonance. **(O.U. Oct. 2001)**

29. (a) Explain the terms resonance, sharpness of resonance and Q-factor.
(b) Set up the equation of motion for forced vibrations and obtain its solution. **(O.U. Oct. 2007)**

30. Define quality factor and bandwidth of the sharpness of resonance. Obtain quality factor for a driven harmonic oscillator at resonance. **(S.V.U. 2007)**

31. (a) Explain forced vibrations. Discuss mathematically the vibrations of a system executing damped S.H.M. when subjected to an external periodic force. Obtain an expression for the amplitude of forced vibrations and discuss the conditions of resonance. **(O.U. 1995)**
(b) Explain the meaning of sharpness of resonance. **(A,U. 1992)**

32. Describe the transient and steady states of a forced oscillator.

33. Write down the differential equation for a forced harmonic oscillator and explain the significance of each term in it. Obtain solution of this equation and discuss it. Write the condition of resonance and explain the sharpness of resonance.

34. For a driven harmonic oscillator of natural frequency $\omega/2\pi$ described by

$$m\ddot{x} + r\dot{x} + kx = F \sin pt,$$

deduce an expression for velocity amplitude x_0. Show that for the case $p >> \omega$, $p = \omega$ and $p << \omega$, this amplitude is governed by inertia factor, the resistance factor and the spring factor respectively.

35. Explain the power absorption, quality factor and resonance of a forced oscillator.

36. Write down the expression for the amplitude of a forced oscillator. Explain the resonance half power points on the amplitude-frequency curve. Show that the quality factor $Q = \frac{\omega_r}{2\Delta\omega}$, where the symbols have their usual meanings.

37. Prove that for the forced oscillations of a damped oscillator the average power of the applied force is equal to the average power dissipated by the damping force.

38. What do you understand by resonance? Explain the sharpness of resonance with the help of theoretical results of forced oscillations and deduce the expression for its quality factor.

39. Show that in a driven oscillator, the maximum power is absorbed at the frequency of velocity resonance and not at the frequency of amplitude resonance.

40. Explain the details of the following terms in case of driven harmonic oscillator:
(i) Quality factor, (ii) Sharpness of resonance, and (iii) Velocity resonance.

Short Answer Questions

1. Drive differential equation of a damped harmonic oscillator. **(A.U. 2014; S.V.U. 2003)**

2. Define damped harmonic motion and write the equation of a damped harmonic oscillator. **(S.K.U. 2014)**

3. Give two examples of damped harmonic oscillations. **(K. U. 2012)**

4. What are relaxation time and quality factor. **(K.U. 2014; S.K.U. 2007)**

5. Form the equation for damped oscillator and obtain its solution. **(A.U. March 2007)**

6. Write the differential equations for the following cases of damped harmonic oscillator: (i) mechanical, (ii) electrical.

[**Hint:** (i) $m\frac{d^2x}{dt^2} + \frac{\gamma dx}{dt} + kx = 0$. (ii) $\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$.]

7. Define damping constant and relaxation time.

8. What is a lograthmic decrement? obtain an expression for it. **(O.U. 2014)**

9. What is lograthmic decrement and how to correct it? **(A.U. 2013; Dr. B.R.A.U. 2013)**

10. The differential equation for damped harmonic oscillator is $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0$

If $\omega_0 > b$, find the time in which amplitude falls to $1/e$ times the initial value.
[**Ans.** $1/b$ sec.]

[**Hint:** $a = a_0 e^{-bt}$ or $\frac{a_0}{a} = \frac{1}{e} = e^{-bt}$ or $e^{-1} = e^{-bt}$ or $bt = 1$ or $t = 1/b$ sec.]

11. The equation of motion for a damped oscillator if given by

$$\frac{d^2x}{dt^2}+\frac{r}{4}\frac{dx}{dt}+8x = 0.$$

For what range of values for the damping constant, will motion be under damped. **(O.U. 2005)**

12. What is meant by '*Q*' of an oscillator? How is it affected by damping? **(O.U. 2005)**

13. The value of quality factor is high for small damping. Explain this statement.

14. Show that the energy of oscillation in a damped harmonic oscillator decreases to e^{-1} of its initial value in the time τ and in this time, the oscillator performs $Q/2\pi$ oscillations.

[**Note:** This is why τ is sometimes called decay time.]

[**Hint:** $E = E_0\, e^{-t/}\tau$. In time $t = \tau$, $\frac{E}{E_0} = e^{-1}$ or $\frac{E}{E_0}=\frac{1}{e}$

$Q = \omega_0\tau$ or $\tau = \frac{Q}{\omega_0}$, $T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0}$ (for low damping)

$\therefore$ Number of oscillations in τ time $= \frac{\tau}{T} = \frac{Q}{\omega_0}\frac{\omega_0}{2\pi}=\frac{Q}{2\pi}$.]

15. Explain what do you mean by overdamped and critically damped vibrations. **(S.V.U. 2007)**

16. What is resonance? Explain with two examples. **(S.VU. 2002)**

17. Discuss the cases of overdamping, underdamping and critical damping. **(A.U. June 2007)**

18. Explain the phenomenon of resonance. What is *Q*-factor? **(A.U. June 2007, 03)**

19. Define quality factor. **(A.U. March 2007)**

20. Explain *Q*-factor. Give an expression for *Q* factor in a mechanical oscillator. **(O.U. 2009; A.N.U. 2008)**

21. Define *Q*-factor and mention is importance. **(O.U. 2001)**

22. Explain how the sharpness of resonance is affected by *Q*-factor. **(A.N.U. March 2007)**

23. Explain the terms quality factor, sharpness of resonance and power factor. **(K.U. 2007)**

24. Define logarithmic dcrement to damped harmonic oscillator. Describe the electrial analoge for a simple oscillator. **(S.K.U. 2003)**

25. Explain amplitude resonance. **(Dr. B.R.A.U. 2014; S.V.U. 2014; A.N.U. 2013, 12; A.U. 2013)**

26. What is resonance? Explain amplitude resonance. **(R.U. 2014; Dr. B.R.A.U. 2013)**

27. Explain *Q*-factor and sharpness of resonance. **(S.U. 2014; M.G.U. 2014)**

28. Explain the term velocity resonance and amplitude resonance. **(S.K.U. 2005)**

29. What is resonance? Explain half power points. **(K.U. 2005)**

30. How are the terms quality factor, band-width and resonance frequency are related. Explain using a diagram. **(O.U. 05)**

31. Show that at resonance for low damping, the phase difference between the force and resultant displacement of the oscillator is $\pi/2$.

32. Explain the electrical analogue of forced oscillator. **(O.U. 2002)**

33. What are transient and steady states? Explain.

PROBLEMS

[Damped Harmonic Oscillator]

1. The time period of pendulum is 2 sec. If the angular retardation due to air is 0.04 time angular velocity of pendulum and original amplitude is 1°, calculate the amplitude after 10 oscillations.

[**Ans.** 40′ approx.]

2. A simple pendulum has a period of 2 sec. and amplitude of 2°. After 10 complete oscillations its amplitude is reduced to 1.5°. Find the relaxation time and damping constant.
[**Ans.** 34.76 sec; 0.01438 sec^{-1}.]

3. The quality factor of a tuning fork is 6×10^4 and its frequency is 512 Hz. In what time its energy will become 1/10 of its initial value?
[**Ans.** 42.94 sec.]

4. A damped vibrating system, starting from rest reaches the first amplitude of 2 cm which reduces to 0.2 cm in that direction after 50 oscillations, each of period 2.3 sec. Find the correction for the first displacement for damping. ($\log_e 10 = 2.3$)
[**Ans.** 0.23 mm]

5. In how much time will the amplitude of a damped harmonic oscillator be reduced to half its initial value, if the decay modulus is one second?
[**Ans.** 0.69 second]

6. The quality factor of a tuning fork of frequency 256 is given by 4×10^4. Calculate the time in which the energy is reduced to $(1/e)^{th}$ of its energy in the absence of damping. How many oscillations does the oscillator make in this time?
[**Ans.** 24.88 sec, 3669.28]

7. The oscillations of a tuning fork of frequency 256 Hz die away to 1/*e* times their amplitude in one second. Show that the frequency due to air damping is practically unaffected.

8. Find the *Q*-factor of a second pendulum if its amplitude falls to one-third of its initial value in 56 oscillations?
[Ans. **160]**

9. The amplitude of an oscillator of frequency 200 Hz falls to 1/10 of its initial value after 2000 cycles. Calculate (i) its relaxation time, (ii) its quality factor, (ii) time in which its energy falls to 1/10 of its initial value, (iv) damping constant.
[**Ans.** (i) 2.174 sec. (ii) $Q = 2730$, (iii) $t = 5$ sec, (iv) $b = 0.23$]

10. The differential equation of an oscillating system is expressed as: $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + n^2x = 0$.

If $b << n$, find the time in which: (i) the amplitude of oscillation falls to 1/*e* of its initial value, (ii) the energy falls to 1/*e* of its initial value.
[**Ans.** (i) (1/*b*) sec. (ii) (1/2*b*) sec.]

11. The amplitude of an oscillator of frequency 200 hertz reduces to 1/10th of its initial amplitude after 2000 oscillations. Calculate the quality factor, relaxation time and damping constant.
[**Ans.** 2.73×10^3; 2.17 sec.]

12. The quality factor of a tuning fork of frequency 384 Hz is 10^4. How much time will be taken in reducing the energy 1/*e* of its initial value? How many oscillations take place in this duration?
[**Ans.** 4.1 sec.; 1574.47.]

13. The quality factor *Q* of a sonometer wire is 2×10^2. On plucking, the wire emits a note of frequency 120 Hz. Find the time in which the emplitude falls to $(1/e^2)$ of the initial value.
[**Ans.** 10.6 sec.]

14. Give the theory of oscillations in series LCR circuit with small damping. Deduce the frequency and quality factor for a circuit with $L = 2$ mh, $C = 5$ μF and $R = 0.2$ ohm.
[**Ans.** 1.59×10^3 Hz; 100.]

15. In a LCR circuit L = 10 mh, C = 1 μF and R = 0.1 ohm. How long a time does the charge oscillation take to decay half amplitude? What is the quality factor for the circuit?
[**Ans.** 0.14 sec; 1000.]

16. In a LCR circuit, L = 0.3 henry and C = 30 μF. Find the maximum value of the resistance which will make the series circuit to be oscillatory.
[**Ans.** 200 Ω]

17. A condenser of 3 microfarad capacity is discharged through a 2 ohm resistance and 4 henry inductance. Show that the discharge is oscillatory. Calculate the frequency and Q-factor of the circuit. Find the time in which the charge amplitude becomes 1% of the initial value.
[**Ans.** The discharge is oscillatory; 18.4 sec.]

[**Hint.:** If $\frac{1}{LC} > \frac{R^2}{4L^2}$, the discharge is oscillatory.]

[Driven Harmonic Oscillator]

1. Write the equation of motion of a simple harmonic undamped oscillator to which a force $F = F_0 \cos pt$ is applied. Verify that its solution is

$$x = \frac{F_0}{m(\omega_0^2 - p^2)} \cos pt.$$

Discuss the resonance in this case.

2. In a forced harmonic oscillator, the amplitude of vibration increases from 0.01 mm at very low frequencies to a maximum value 5 mm at the frequency 100 c.p.s. Calculate the quality factor, damping constant and half width of the resonance curve.
[**Ans.** 500; 1.25/sec.; 2.2 rad./sec.]

3. If Q = 50, find the values of A/A_{max}, when p/ω_0 = 0.98 and 0.97.
[**Ans.** 0.45, 0.32.]

4. A mass of 1 kg is suspended from a linear spring with a force constant $C = 10^3$ N/m, and damping coefficient $\gamma = 50 \times 10^{-3}$ N-sec/m. The spring is driven by an external force $F = F_0 \sin pt$, where F_0 = 2.5 newton and p is twice the natural frequency of the system. (i) Calculate the amplitude of the resulting motion. (ii) How much is the displacement shifted in phase from the driving force?
[**Ans.** (i) 8.3×10^{-4} m; (ii) –179.9º.]

5. For a high Q-system, show that the width of the amplitude resonance curve is nearly $\sqrt{3}b$, where the full-width is measured between those frequencies where $a = a_m/2$.

6. Calculate the average energy stored in a 20 gm mass attached to a spring and vibrating with an amplitude 1 cm in resonance with a periodic force whose frequency is 20 Hz. If the quality factor of the oscillator be 160, how much energy being dissipated per second?
[**Ans.** 0.016 J; 0.0125 J.]

7. A condenser of 0.5 μF capacity is discharged through a 2 ohm resistance and a 2 henry inductance. Find the freqency of electrical oscillations and the time in which the amplitude of oscillations is reduced to 0.1%.
[**Ans.** 160 Hz; 14 sec.]
[**Hint.:** $Q_0/1000 = Q_0 e^{-Rt/2L} = Q_0 e^{-t/2}$ or $e^{t/2} = 1000$.]

8. A torsional pendulum with moment of inertia 900 gm-cm^2 and torsion constant 4×10^{-3} N-m/rad executes resonant vibrations under the influence of an external torque. Calculate the frequency of the applied torque. What should be the Q of the oscillator so that the amplitude of oscillation may reduce to 20%, when the frequency of the applied torque increased by 0.2%?
[**Ans.** 1.06 Hz; 187.]

9. A circuit contains a resistance of 2 ohm, an inductance of 0.005 henry and a capacity of 600 μF in series with a potential difference of 100 volts at driven frequency 50 Hz. Find the current, the power factor and the power.

[**Ans.** 26.5 amp (r.m.s value) ahead of the driven voltage by $61^{o}54'$; $\cos\phi = 0.471$; $p = 1114$ watts.]

10. In an experiment on forced oscillations the frequency of a sinusoidal driving force is changed while its amplitude is kept constant. It is found that the amplitude of vibration is 0.10 mm at very low frequency of the driver and goes upto a maximum of 0.5 mm at driving frequency 200 sec^{-1}. Calculate (i) Q of the system, (ii) relaxation time τ, and (iii) half width of resonance curve.

[**Ans.** (i) 500; (ii) 0.4 sec; (iii) 2.16 rad/sec.]

11. Calculating the resonating frequency and Q of the circuit containing 0.02 μF capacity, 80 mh inductance and 0.25 ohm resistance.

[**Ans.** 4×10^3 Hz; 8×10^3.]

12. Find the resonating frequency and Q of a circuit containing 0.04 μF capacity, 40 mH inductance and 0.2 ohm resistance. What is the band-width of the power resonance curve?

[**Ans.** $\nu_0 = 3.98 \times 10$ Hz; $Q = 5000$; $\Delta p = 5$ rad./sec.]

13. A condenser of 0.5 μF capacity is discharged through a 2 ohm resistance and a 2 henry inductance. Find the frequency of electrical oscillations and the time in which the amplitude of oscillations is reduced to 0.1%.

[**Ans.** 160 Hz; 14 sec..]

14. A resistance of 10 ohms is joined in series with an inductance of 0.5 henry. What capacitance should be put in series with the combination to obtain maximum current? What will be the potential difference across the resistance, inductance and capacity? The current is being supplied by a 200 volts and 50 cycles A.C. mains.

[**Ans.** 20.27×10^{-6} farad; 20 amp; 200 volts; 3141.6 volts, 3141.6 volts.]

15. A coil of 3 mH is given. Calculate the capacity which will make the resonance frequency 1000 kilo Hz.

[**Ans.** 8.85 peko-farad.]

16. LCR circuit has $R = 10$ ohms, $L = 200$ micro-henry and $C = 500$ peko-farad. Determine the angular frequency separation between the half power points and quality factor for the circuit.

[**Ans.** 5×10^4 rad./sec.; 63.3.]

●

3

Complex Vibrations

1. Periodic Function

Any function which repeats itself regularly over a given interval of time or space is called a periodic function. Mathematicaly, a function $f(t)$ is said to be periodic if for all time t, we have

$$f(t + T) = f(t)$$

where T is a constant, calle the period. For example, sinusoidal functions, sin ωt and cos ω*t, are periodic* functions of period $T = 2\pi/\omega$ because

$$\sin\omega\left(t+\frac{2\pi}{\omega}\right) = \sin\ \omega t \text{ and } \cos\omega\left(t+\frac{2\pi}{\omega}\right) = \cos\ \omega t$$

There may be functions of time t which are periodic but not sinusoidal. A periodic saw-tooth wave form, used in the time base of a cathode ray oscillograph, a square wave generated by an electronic function generator etc. are examples of such periodic functions.

A simple harmonic vibration is the simplest type of vibration. When few such simple harmonic vibrations are superimposed, we obtain a periodic vibration or a periodic function of complex nature. For example, the vibrations represented by saw-tooth wave or square wave is a complex periodic function. Although these functions are quite simple to visualize, but they are very complicated to represent mathematically. Fourier theorem deals with the analysis of such complex periodic functions in terms of simple harmonic components.

2. Fourier's Theorem

In 1822, Fourier stated a very important theorem, with the help of which any complex periodic function can be analysed. According to this theorem, *any finite single valued periodic function, whatever, can be expressed as a summation of simple harmonic terms which have frequencies, being integral multiples of that of the given function.* The series, so obtained, is called *Fourier's series.*

Mathematically Fourier's theorem can be represented by the series

$$y = f(\omega t) = A_0 + A_1 \sin\omega t + A_2 \sin 2\omega t + \ldots + A_r \sin r\,\omega t + \ldots + B_1 \cos\omega t + B_2 \cos 2\omega t + \ldots + B_r \cos r\omega t + \ldots \quad \ldots(1)$$

or

$$y = f(\omega t) = A_0 + \sum_{r=1}^{\infty} A_r \sin r\omega t + \sum_{r=1}^{\infty} B_r \cos r\omega t \quad \ldots(2)$$

where $y = f(\omega t)$ is the displacement of complex periodic vibration of frequency $\omega/2\pi$ at any instant t and r is a positive integer *i.e.*, $r = 1, 2, 3, \ldots$. Here the periodic function has been expressed as the summation of sine and cosine components, which have frequencies, multiples of $\omega/2\pi$.

Evaluation of the values of A_0, A_1, ...A_r and B_1, B_2, ...B_r.

To find A_0, multiply equation (1) by dt and integrate for complete vibration, i.e., from 0 to T $(= 2\pi/\omega)$, we get

---

* $\int_0^T \sin r\,\omega t\,dt = \left[-\frac{\cos r\,\omega t}{r\omega}\right]^{T=2\pi/\omega} = \frac{-\cos 2\pi r + \cos 0}{r\omega} = 0$ and $\int_0^T \cos r\,\omega t\,dt = \left[\frac{\sin r\omega t}{r\omega}\right]_0^{T=2\pi/\omega} = \frac{\sin 2\pi r - \sin 0}{r\omega} = 0$

$$\int_0^T y\,dt = A_0\int_0^T dt + A_1\int_0^T \sin\omega dt + \ldots + A_r\int_0^T \sin r\omega t\,dt$$
$$+ \ldots + B_1\int_0^T \cos\omega t\,dt + \ldots + B_r\int_0^T \cos r\omega t\,dt + \ldots$$
$$= A_0 T$$
$$\therefore \quad A_0 = \frac{1}{T}\int_0^T y\,dt \qquad \ldots(3)^*$$

To find the value of A_r, multiply equation (1) by $\sin r\omega t\,dt$ and integrate for complete vibration. Then we get

$$\int_0^T y\sin r\omega t\,dt = A_0\int_0^T \sin r\omega t\,dt + A_1\int_0^T \sin\omega t\sin r\omega t\,dt + \ldots + A_r\int_0^T \sin^2 r\omega t\,dt + \ldots$$
$$+ B_1\int_0^T \sin r\omega t\cos\omega t\,dt \ldots + B_r\int_0^T \sin r\omega t\cos r\omega t\,dt + \ldots$$
$$= \frac{A_r}{2}\int_0^T (1-\cos 2r\omega t)\,dt = A_r \times \frac{T}{2}$$
$$\therefore \quad A_r = \frac{2}{T}\int_0^T y\sin r\omega t\,dt = \frac{2}{T}\int_0^T y\sin\frac{2\pi rt}{T}dt \qquad \ldots(4)$$

Similalry values of B_r can be found by multiplying eq. (1) by $\cos r\omega t\,dt$ and then on integrating we will get

$$B_r = \frac{2}{T}\int_0^T y\cos r\omega t\,dt = \frac{2}{T}\int_0^T y\cos\frac{2\pi rt}{T}dt \qquad \ldots(5)$$

Thus if we put r = 1, 2, 3, ... in eq. (4) and (5) we can calculate the coefficients A_1, A_2 ...., B_1, B_2, ... of sine and cosine terms. Hence we can analyse $y = f(\omega t)$, i.e., the displacement of complex periodic vibrations in terms of sine and cosine terms.

Significance of the Fourier constant A_0

The constant A_0 of the Fourier's series is evaluated by using the relation (3) i.e.,

$$A_0 = \frac{1}{T}\int_0^T y\,dt = \frac{1}{T}\int_0^T f(\omega t)dt$$

Thus A_0 is the average value of the function y or $f(\omega t)$ over the period T and represents the mean displcement for a complete period from the X-axis. The term A_0 denotes the distance of the axis of symmetry of the diplacement-time curve of the complex vibration from the axis of coordinates or time axis. Obviously the straight line $y = A_0$ (constant) represents the axis of the complex vibration curve. In a period of complex vibration, if for half the period the function y [or $f(\omega t)$] has positive values and for the other half of the period, y has equal negative values, then the value of A_0 is zero.

For a square wave, when the axis of vibration is at a distance $a/2$ from X-axis, the value of A_0 is is $a/2$ **[Fig. 1 (i)]** and when it coincides with X-axis, the value of A_0 is zero **[Fig. 1 (ii)]**; in both figures the displacement y has been measured from X-axis or time axis.

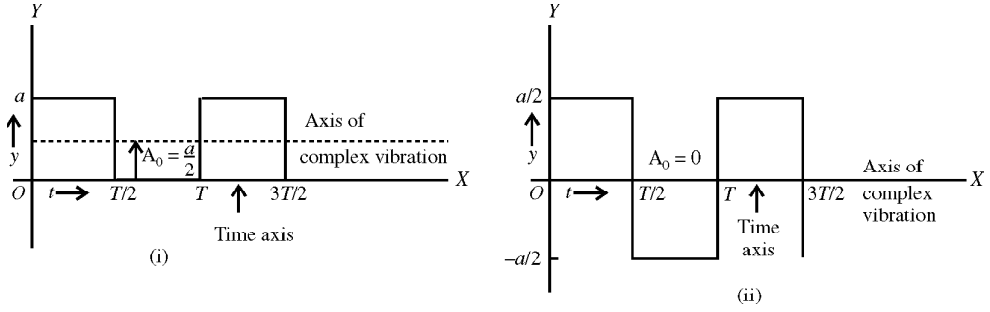


Fig. 1: Representing the value of A_0 for a square wave to be (i) $a/2$, when axis of vibration is at a distance of $a/2$ from X-axis and (ii) zero, when both axes coincide.

Fourier series for a periodic function of variable *x*

In case, if $y = f(x)$ is a periodic function of variable x of period L, then

$$A_0 = \frac{1}{L}\int_0^L f(x)dx; A_r = \frac{2}{L}\int_0^L f(x)\sin\frac{2\pi rx}{L}\,dx \text{ and } B_r = \frac{2}{L}\int_0^L f(x)\cos\frac{2\pi rx}{L}\,dx \quad ...(6)$$

and the series is

$$y = f(x) = A_0 + \sum_{r=1}^{\infty} A_r \sin\frac{2\pi rx}{L}dx + \sum_{r=1}^{\infty} B_r \cos\frac{2\pi rx}{L}dx \quad ...(7)$$

If the periodic function $f(x)$ has a period 2π in the interval $-\pi$ to π, it can be expressed as a Fourier's series of sines and cosines in the form:

$$f(x) = A_0 + \sum_{r=1}^{\infty} A_r \sin rx + \sum_{r=1}^{\infty} B_r \cos rx \quad ...(8)$$

where the coefficients are evaluated as

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx; A_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin rx\,dx \text{ and } B_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx\,ds \quad ...(9)$$

3. Limitations of Fourier's Theorem

In order to aply fourier's theorem to practical problems, the theorem is subjected to the following limitations regarding the form of complex vibration:

(1) The function should be periodic single-valued and continuous: This means that the displacement should repeat after an interval of time and it must be continuous with only one value at an instant. In other words, the displacement should not have two or more values at the same time. In **fig. 9.2** at the point *A*, the function is not single-valued but the particle has two displacements at the same instant. In such a case, Fourier's theorem is not applicable.

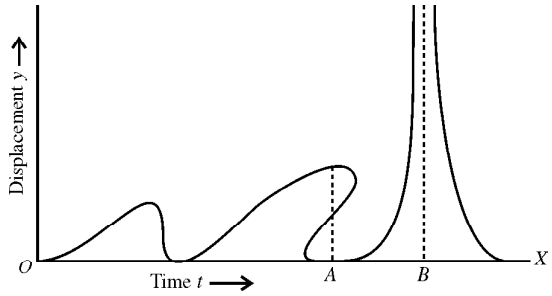


Fig. 2: Discussing limitations of faurier's theorem.

(2) The function should always be finite: This means that the displacement during oscillations should always remain finite and it should not be infinite at any time as at B **(fig. 9.2)**.

(3) Moreover, the function should have a finite number of discontinuities and a finite number of maxima and minima in the interval of one oscillation, *T*.

4. Applications of Fourier's Theorem

The Fourier's analysis in applicable to periodic functions of time as well as those of space. Here, we shall apply the Fourier's theroem to analyse the following complex periodic vibrations:

(a) Square wave

(b) Saw-tooth wave

(c) Triangular wave

These periodic curves cannot be represented by pure sine or cosine curves, but we shall see that they can be built up by compounding a number of sinusoidal curves.

(a) Fourier Analysis of Square Wave

Let us consider the Fourier analysis of a square wave as shown in **fig. 3.** We represent the periodic function $y = f(\omega t)$, such as displacement, along *Y*-axis and time along *X*-axis. Here, *y* remains constant from time $t = 0$ to $t = T/2$ and then remains zero for the half period. Hence we may express this function as follows:

$$y = a, \quad t = 0 \quad \text{to} \quad t = T/2$$
$$y = 0, \quad t = T/2 \text{ to } t = T. \qquad ...(1)$$

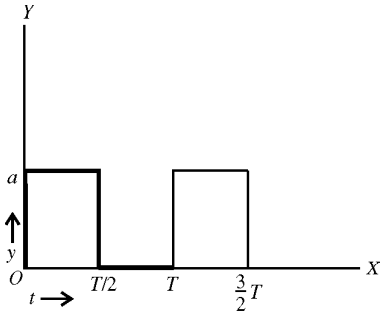


Fig. 3: Square wave.

We have to find the Fourier's series of this function. According to Fourier's theorem, we have

$$y = A_0 + A_1 \sin \omega t + A_2 \text{ sing } 2\ \omega t + ...A_r \sin r\omega t + ... + B_1 \cos \omega t + ... + B_r \cos r\omega t + ... \qquad ...(2)$$

$$\therefore \quad A_0 = \frac{1}{T}\int_0^T y\,dt = \frac{1}{T}\int_0^{T/2} a\,dt + 0$$

$$= \frac{a}{T} \times \frac{T}{2} = \frac{a}{2} \qquad ...(3)$$

$$A_r = \frac{1}{T}\int_0^{T/2} y \sin r\omega t\ dt + 0 = \frac{2}{T}\int_0^{T/2} a \sin r\omega t\ dt$$

$$= \frac{2a}{T}\left[-\frac{1}{r\omega}\cos r\omega t\right]_0^{T/2} \qquad \left[\because \omega = \frac{2\pi}{T}\right]$$

$$= \frac{2a}{T} \times \frac{T}{2\pi r}(-\cos r\pi + 1) = \frac{a}{\pi r}(1 - \cos r\pi) \qquad ...(4)$$

Now, when $r = 0, 2, 4...A_r = 0$ ...(5a)

and when $r = 1, 3, 5... A_r = \frac{2a}{\pi r}$ ...(5b)

$$B_r = \frac{2}{T}\int_0^{T/2} a \cos r\omega t\ dt$$

$$= \frac{2}{T}\left[\frac{a}{r\omega}\sin r\omega t\right]_0^{T/2} \qquad \left[\text{since } \omega = \frac{2\pi}{T}\right]$$

$$= \frac{2}{T}\left[\frac{a}{r\omega}\left(\sin r \times \frac{2\pi}{T} \times \frac{T}{2} - 0\right)\right] = \frac{2a}{Tr\omega}(\sin r\pi - 1) = 0 \qquad ...(6)$$

Therefore, substituting the values of A_0, A_r and B_r in eq. (2) from (3), (5) and (6), we obtain the Fourier's series for the given function as

$$y = \frac{a}{2} + \frac{2a}{\pi}\left[\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + ... + \frac{1}{r}\sin r\omega t + ...\right] \qquad ...(7)$$

We see that the complex perioic function is represented as the superposition of a number of simple harmonic motions **[fig. 4]**. The frequencies of these component motions are an odd multiple of the fundamental frequency, i.e., in this case only odd harmonics are present. The amplitude of the successive harmonics is proportional to 1, $\frac{1}{3}$, $\frac{1}{5}$,..., $\frac{1}{r}$,.... This is the characteristic property of a particular wave form and is independent of the instant for measuring the time. Expression (7) contains only sine terms which depend on the instant from which the time is measured. If the time is measured $T/4$ second later

than in **fig. 3**, then the curve will be symmetrical about *Y*-axis and there will be only cosine terms in the series (see below **Ex. 3**).

This is also clear that A_0 (here equal to $a/2$) is the average value of y. This means that the axis of the curve is at a distance $a/2$ up the time axis. In case, if for equal intervals of time, y has equal positive and negative values, the value A_0 is zero (see below **Ex. 1**) and the axis of the curve is the time axis.

Taking four terms (constant and first three harmonics) of the Fourier's series [eq. (7)], resultant curve is shown in **fig. 4** by a thick line. As more highr harmonics are added, the curve more nearly approaches the original, as shown in **fig. 3**.

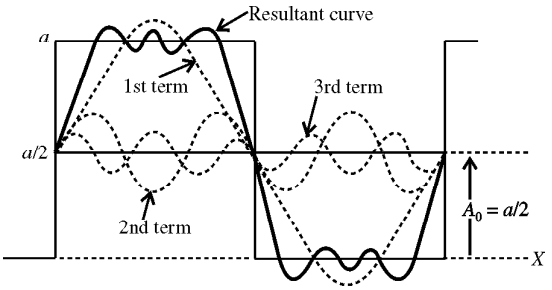


Fig. 4: Resultant of first three sine erms and the constant $A_0 = a/2$ of the Fourier's series [eq. 7].

Ex. 1. *Find a Fourier's series for the function defined by the equations*:

$$y = \begin{cases} +a & for\, t = 0\, to\, t = T/2 \\ 0 & for\, t = T/2 \\ -a & for\, t = T/2\, to\, t = T \end{cases}$$

Sol. The given function, a square wave, is shown is **fig. 5**. Let the Fourier's expansion be given by

$$y = A_0 + A_1 \text{ sig } \omega t + ... + A_r \sin r\omega t + ... + B_1 \cos \omega t + ... + B_r \cos r\omega t + ... \quad ...(1)$$

where $\omega = 2\pi/T$.

$$A_0 = \frac{1}{T}\int_0^T y\, dt$$

$$= \frac{1}{T}\int_0^{T/2}(+a)\, dt + 0 + \frac{1}{T}\int_{T/2}^{T}(-a)\, dt$$

$$= \frac{a}{2} - \frac{a}{2} = 0 \qquad ...(2)$$

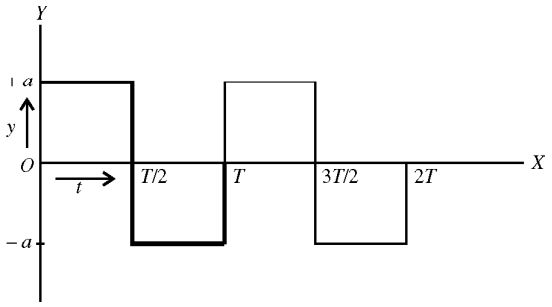


Fig. 5 : Squrare wave.

$$A_r = \frac{2}{T}\int_0^T y \sin r\omega t\, dt$$

$$= \frac{2}{T}\int_0^{T/2} a \sin r\omega t\, dt + 0 + \frac{2}{T}\int_{T/2}^{T}(-a)\sin r\omega t\, dt$$

$$= \frac{2a}{T}\left[-\frac{\cos r\omega t}{r\omega}\right]_0^{T/2} - \frac{2a}{T}\left[-\frac{\cos r\omega t}{r\omega}\right]_{T/2}^{T} \qquad \left[\because\ \omega = \frac{2\pi}{T}\right]$$

$$= \frac{2a}{Tr\omega}\left[-\cos r \times \frac{2\pi}{T} \times \frac{T}{2} + 1\right] - \frac{2a}{Tr\omega}\left[-\cos r \times \frac{2\pi}{T} \times T + \cos r \times \frac{2\pi}{T} \times \frac{T}{2}\right]$$

$$= \frac{a}{r\pi}\left[-\cos r\pi + 1\right] + \frac{a}{r\pi}\left[1 - \cos r\pi\right]$$

$$= \frac{2a}{r\pi}\left[1 - \cos r\pi\right]$$

= 0, when r is even i.e., r = 2, 4, 6, ...(3a)

$= \frac{4a}{r\pi}$, when r is odd i.e., r = 1, 3, 5, ...(3b)

and $B_r = \frac{2}{T}\int_0^{T/2} a \cos r\omega t\, dt + \frac{2}{T}\int_{T/2}^{T}(-a)\cos r\omega t\, dt$

$= \frac{2a}{T}\left[\frac{\sin r\omega t}{r\omega}\right]_0^{T/2} - \frac{2a}{T}\left[\frac{\sin r\omega t}{r\omega}\right]_{T/2}^{T}$ $\left[\text{since } \omega = \frac{2\pi}{T}\right]$

$= \frac{a}{r\pi}[\sin r\pi - 0] - \frac{a}{r\pi}$ [sin $2r\pi$ – sing $r\pi$]

$= \frac{a}{r\pi}[0] - \frac{a}{r\pi}[0] = 0$ ...(4)

Substituting for A_0, A_r and B_r from (2), (3) and (4) in (1), we get the Fourier's series as

$$y = \frac{4a}{\pi}\sin\omega t + \frac{4a}{3\pi}\sin 3\omega t + \frac{4a}{5\pi}\sin 5\omega t + ... + \frac{4a}{r\pi}\sin r\omega t + ...$$

$$y = \frac{4a}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + ... + \frac{1}{\pi}\sin r\omega t + ...\right] \quad ...(5)$$

Ex. 2. *Find the Fourier's series for the function f(t) defined by*

$f(t) = 0$ *for* $0 < t < T./2$.

$f(t) = 1$ *for* $T/2 < t < T$ *and* $f(t + T) = f(t)$. **(O.U. March 2007)**

Sol. *In the given case,*

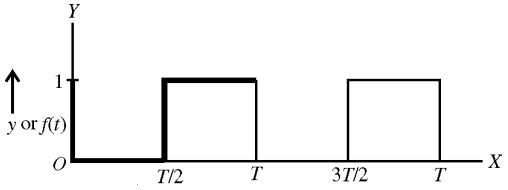


Fig. 6: Square wave.

$y = f(t) = 0$ for $0 < t < T/2$

and $y = f(t) = 1$ for $T/2 < t < T$ ...(1)

also $f(t + T) = f(t)$.

This means that $f(t)$ is a periodic function. The given function, a square wave, is represented in **fig. 6**. Let the Fourier's series be given by

$$y = A_0 + A_1 \sin\omega t + A_2 \sin 2\omega t + ... + A_r \sin r\omega t + ... + B_1 \cos\omega t + ... + B_r \cos r\omega t + ... \quad ...(2)$$

Fourier coefficients are evaluated as follows:

$$A_0 = \frac{1}{T}\int_0^T y\,dt = \frac{1}{T}\int_0^{T/2} 0\times dt + \frac{1}{T}\int_{T/2}^{T} 1\times dt = \frac{1}{T}[t]_{T/2}^{T} = \frac{1}{2} \quad ...(3)$$

$$A_r = \frac{2}{T}\left[\int_0^{T/2} y\sin r\omega t\, dt + \int_{T/2}^{T} y\sin r\omega t\, dt\right]$$

$$= 0 + \frac{2}{T}\int_{T/2=\pi/\omega}^{T=2\pi/\omega}\sin r\omega t\, dt = \frac{2}{2\pi/\omega}\left[-\frac{\cos r\omega t}{r\omega}\right]_{\pi/\omega}^{2\pi/\omega}$$

$$= \frac{\omega}{\pi}\times\frac{1}{r\omega}[-\cos 2r\pi + \cos r\pi]$$

$$= \frac{1}{r\pi}(\cos r\pi - 1) \quad ...(4)$$

and $$B_r = \frac{2}{T}\left[\int_0^{T/2} y\cos r\omega t\,dt + \int_{T/2}^{T} y\cos r\omega t\,dt\right]$$

$$= 0 + \frac{2}{T}\int_{T/2=\pi/\omega}^{T=2\pi/\omega} \cos r\omega t\,dt = \frac{2}{2\pi/\omega}\left[-\frac{\sin r\omega t}{r\omega}\right]_{\pi/\omega}^{2\pi/\omega} = 0 \quad ...(5)$$

When r is even, $\cos 2r\pi = 1$ and when r is odd i.e., $r = 1, 3, 5, ..., \cos r\pi = -1$. This means from (4)

$$A_r = 0 \qquad \text{for even } r$$

$$A_r = -\frac{2}{r\pi} \qquad \text{for odd } r \quad ...(5)$$

Hence from (2), the Fourier's series for the given periodic function is

$$y = f(t) = \frac{1}{2} - \frac{2}{\pi}\sin\omega t - \frac{2}{3\pi}\sin 3\omega t ..... - \frac{2}{r\pi}\sin r\omega t .....$$

or $$y = f(t) = \frac{1}{2} - \frac{2}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 2\omega t + ... + \frac{1}{\pi}\sin r\omega t + ...\right]$$

for add r ...(6)

Ex. 3. *Find the Fourier's coefficients for the function, represented by the following equations:*

$$y = 0,\ -\frac{T}{2} < t < -\frac{T}{4} \quad ...(i)$$

$$y = a,\ -\frac{T}{4} < t < +\frac{T}{4} \quad ...(ii)$$

$$y = 0,\ \frac{T}{2} < t < \frac{T}{2} \quad ...(iii)$$

Write the fourier series also.

Sol. The given function is represented in **fig. 7**. Let the Fourier series be given by

$y = A_0 + A_1 \sin \omega t + A_2 \text{ sing } 2\omega t + ... + A_r \sin r\omega t + ... + B_1 \cos \omega t + ... + B_r \cos r\omega t + ...$

Fourier coeficients for the given function are evaluated as follows:

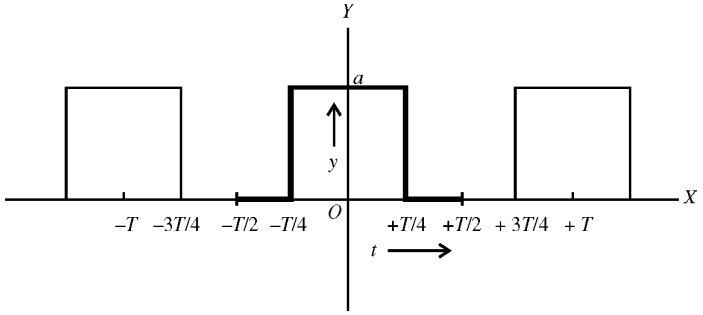


Fig. 7: Square wave.

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} y\,dt = \frac{1}{T}\left[\int_{-T/2}^{T/2} y\,dt + \int_{-T/4}^{+T/4} y\,dt + \int_{T/4}^{T/2} y\,dt\right]$$

$$= \frac{1}{T}\int_{-T/4}^{T/4} a\,dt = \frac{a}{T}\left[\frac{T}{4} + \frac{T}{4}\right] = \frac{a}{2}$$

$$A_r = \frac{2}{T}\int_{-T/2}^{T/2} y\sin r\omega t\ dt = \frac{2}{T}\int_{-T/4}^{+T/4} a\sin r\omega t\ dt = \frac{2a}{T}\left[\frac{-\cos r\omega t}{r\omega}\right]_{-T/4}^{+T/4}$$

$$= \frac{2a}{T}\left[-\frac{\cos\frac{2\pi}{T}rt}{2\pi r/T}\right]_{-T/4}^{+T/4} = \frac{2a}{T}\left[\frac{\cos\frac{r\pi}{2}-\cos r\pi/2}{2\pi r/T}\right] = 0 \qquad \left(\because \omega = \frac{2\pi}{T}\right)$$

and $$B_r = \frac{2}{T}\int_{-T/2}^{T/2} y\cos r\omega t\, dt = \frac{2}{T}\int_{-T/4}^{T/4} a\cos r\omega t\, dt$$

$$= \frac{2}{T}\int_{-T/2}^{T/2} y\cos\omega t\, dt = \frac{2a}{T}\left[\frac{\sin r\omega t}{r\omega}\right]_{-T/4}^{T/4}$$

$$= \frac{2a}{T}\times\frac{T}{2\pi r}\times 2\sin\left(r\times\frac{2\pi}{T}\times\frac{T}{4}\right) = \frac{2a}{r\pi}\sin\left(\frac{r\pi}{2}\right)$$

Thus the Fourier coefficients are

$$A_0 = \frac{a}{2},\ A_r = 0 \text{ and } B_r = \frac{2a}{r\pi}\sin\left(\frac{r\pi}{2}\right)$$

Thus the Fourier series is

$$y = \frac{a}{2} + \frac{2a}{\pi}\left[\cos\omega t + \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t + ...\right]$$

(b) Fourier Analysis of Saw Tooth Wave

A saw-tooth wave, representing the displacement-time durve of a certain oscillatory system, is represented in **fig. 8.** For a saw-tooth wave, we see that when $t = 0$, $y = a$ and then the displacement decreases linearly with time and when $t = T$, $y = 0$. The curve repeats itself after $t = T$.

Now let us take a point P on the linear portion of the saw-tooth curve with coordinates (t, y). From the similar triangles AOB and $PP'B$, we have

$$\frac{a}{y} = \frac{T}{T-t} \text{ or } y = a\frac{(T-t)}{T}$$

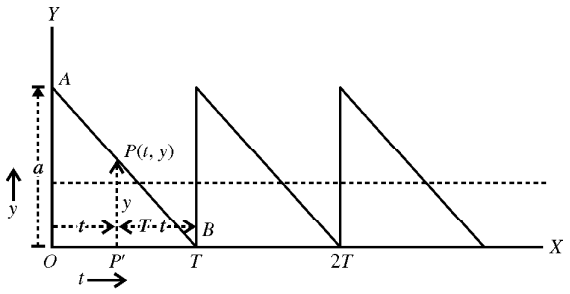


Fig. 8: Saw-Tooth wave.

Thus the saw-tooth wave form can be represented as

$$y = \frac{a}{T}(T-t) \quad \text{for } 0 < t < T \qquad ...(1)$$

The Fourier series is siven by

$$y = A_0 + A_1 \text{ sing } \omega t + A_2 \text{ sing } 2\omega t + ... + A_r \text{ sing } r\omega t + ...$$
$$+ B_1 \cos\omega t + ... + B_r \cos r\omega t + ... \qquad ...(2)$$

Fourier coefficients of the series for the given wave form are determined as follows:

$$A_0 = \frac{1}{T}\int_0^T y\, dt = \frac{a}{T}\int_0^T\left(1-\frac{t}{T}\right)dt = \frac{a}{T}\left[t-\frac{t^2}{2T}\right]_0^T = \frac{a}{T}\left[T-\frac{T^2}{2T}\right] = \frac{a}{2} \qquad ...(3)$$

Hence the axis of curve is at a distance $a/2$ up the time axis.

$$A_r = \frac{2}{T}\int_0^T y\sin r\omega t\, dt = \frac{2a}{T}\int_0^T\left(1-\frac{t}{T}\right)\sin r\omega t\, dt$$

$$= \frac{2a}{T}\int_0^T \sin r\omega t\, dt - \frac{2a}{T^2}\int_0^T t \sin r\omega t\, dt$$

$$= \frac{2a}{T}\left\{-\frac{\cos r\omega t}{r\omega}\right\}_0^T - \frac{2a}{T^2}\left\{\left(-t\frac{\cos r\omega t}{r\omega}\right)_0^T + \int_0^T \frac{\cos r\omega t}{r\omega} dt\right\}$$

(Integrating by parts)

$$= \frac{2a}{Tr\omega}\{-\cos r\omega T + \cos(r\omega \times 0)\} + \frac{2a}{T^2}\cdot\frac{T\cos r\omega T}{r\omega} - \frac{2a}{T^2}\left(\frac{\sin r\omega t}{r^2\omega^2}\right)_0^T$$

Now, $-\cos(r\omega T) + \cos 0 = -\cos(r\omega .\ 2\pi/\omega) + 1 = -\cos 2\pi r + 1 = -1 + 1 = 0$

and $\left(\sin r\omega t\right)_0^T = 0$ $\left[\because T = \frac{2\pi}{\omega}\right]$

$$\therefore \quad A_r = \frac{2a}{T}\cdot\frac{\cos(2\pi r)}{r\omega} = \frac{2a}{Tr\omega} = \frac{2a\omega}{2\pi r\omega} = \frac{a}{r\pi} \qquad ...(4)$$

and $$B_r = \frac{2}{T}\int_0^T y\cos r\omega t\, dt = \frac{2a}{T}\int_0^T \left(1 - \frac{t}{T}\right) \cos r\omega t\, dt = 0 \qquad ...(5)$$

Hence all the cosine terms of Fourier's series are absent.

Hence, the Fourier's series in this case is

$$y = \frac{a}{2} + \frac{a}{\pi}\left(\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + ... + \frac{1}{r}\sin r\omega t + ...\right) \qquad ...(6)$$

We note that in the series all the harmonics are present, having amplitude in the ratio $1 : \frac{1}{2} : \frac{1}{3} :$ ... and frequencies in the ratio 1: 2: 3: ..... Hence the value of the amplitude decreases for higher harmonics.

The addition of successive terms is represented in **fig 9**. We see that the resultant curve tends to resemble the saw-tooth wave form as we include more and more terms inside the bracket.

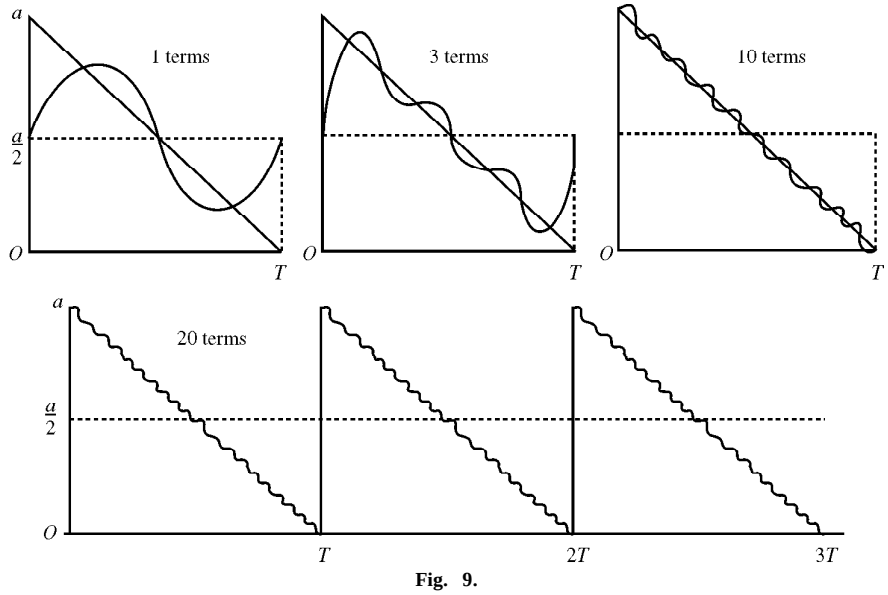


Fig. 9.

(c) Fourier Analysis of Triangular wave

Consider a triangular part *OAB* of the wave **(fig. 10)**. On the linear part of the wave, let us take a point *P* with coordinates (*t, y*). In similar triangles *OAC* and *OPP '*, we have

$\frac{AC}{OC}=\frac{PP'}{OP'}$ or $\frac{a}{T/2}=\frac{y}{t}$ or $y=\frac{2at}{T}$ for $0 < t < T/2$

If we take a point P on AB straight line, then similarly

$\frac{a}{T-T/2}=\frac{y}{T-y}$ or $\frac{a}{T/2}=\frac{y}{T-t}$ or $y=\frac{2a(T-t)}{T}$ for $T/2<t<T$

Hence the displacement at any time t for the triangular wave can be represented as

$$y=\frac{2at}{T} \qquad \text{for } 0 < t < T/2$$

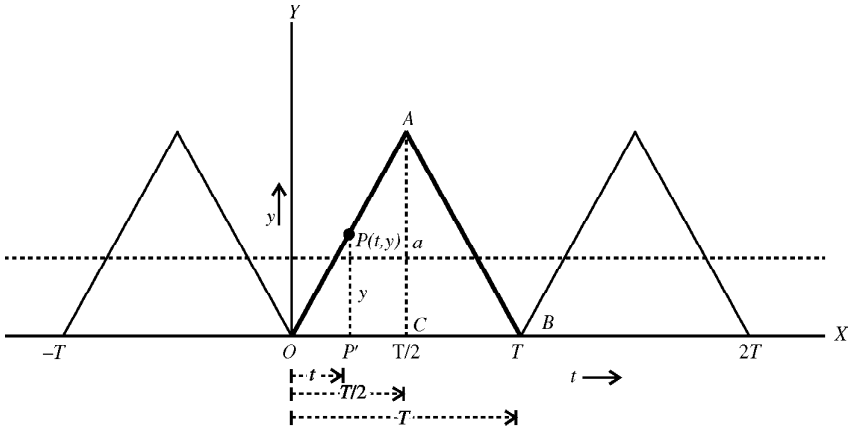


Fig. 10: Triangular wave.

and $$y=\frac{2a(T-t)}{T} \qquad \text{for } T/2 < t < T. \qquad ...(1)$$

After $t = T$ the curve repeats itself.

The Fourier series is given by

$y = f(\omega t)\ A_0 + A_1 \sin \omega t + A_2 \sin 2\omega t + ... + A_r \sin r\omega t + ... + B_1 \cos \omega t + ... + B_r \cos r\omega t + ...$ ...(2)

Now, let us calculate the Fourier coefficients:

$$A_0=\frac{1}{T}\int_0^T y\,dt = \frac{1}{T}\left[\int_0^{T/2}\frac{2at}{T}dt+\int_{T/2}^{T}\frac{2a(T-t)}{T}dt\right]$$

$$=\frac{1}{T}\left[\frac{2a}{T}\left(\frac{t^2}{2}\right)_0^{T/2}+\frac{2a}{T}\left\{T\,t-\frac{t^2}{2}\right\}_{T/2}^{T}\right]=\frac{2a}{T^2}\left[\frac{T^2}{8}+\left(T^2-\frac{T^2}{2}\right)-\left(\frac{T^2}{2}-\frac{T^2}{8}\right)\right]$$

$$=\frac{a}{2} \qquad ...(3)$$

This $A_0 = a/2$ represents the displacement of the axis of the curve from the time axis.

$$A_r=\frac{2}{T}\int_0^T y\sin r\omega t\,dt$$

$$=\frac{2}{T}\left[\int_0^{T/2}\frac{2at}{T}\sin r\omega t\,dt+\int_{T/2}^{T}\frac{2a(T-t)}{T}\sin r\omega t\,dt\right]$$

$$=\frac{4a}{T^2}\left[\int_0^{T/2}t\sin r\omega t\,dt+T\int_{T/2}^{T}\sin r\omega t\,dt-\int_{T/2}^{T}t\sin r\omega t\,dt\right] \qquad ...(4)$$

Now, let us evaluate the value of the integrals in the bracket, taking $T = 2\pi/\omega$.

$$I_1 = \int_0^{T/2=\pi/\omega} t\sin r\omega t\ dt = \left[-t\frac{\cos r\omega t}{r\omega}\right]_0^{\pi/\omega} - \int_0^{\pi/\omega} -\frac{\cos r\omega t}{r\omega}dt$$

$$= -\frac{\pi}{\omega}\frac{\cos r\pi}{r\omega} + \frac{1}{r\omega}\left[\frac{\sin r\omega t}{r\omega}\right]_0^{\pi/\omega} = -\frac{\pi\cos r\pi}{r\omega^2} \quad ...(5)$$

$$I_2 = T\int_{T/2}^{T}\sin r\omega t\ dt = \frac{2\pi}{\omega}\left[-\frac{\cos r\omega t}{r\omega}\right]_{\pi/\omega}^{2\pi/\omega} = \frac{2\pi(\cos r\pi - 1)}{r\omega^2} \quad ...(6)$$

and $$I_3 = \int_{T/2}^{T} t\sin r\omega t\ dt = \left[-t\frac{\cos r\omega t}{r\omega}\right]_{\pi/\omega}^{2\pi/\omega} + \int_{\pi/\omega}^{2\pi/\omega}\frac{\cos r\omega t}{r\omega}dt$$

$$= -\frac{2\pi}{\omega\times r\omega} + \frac{\pi}{\omega}\frac{\cos r\pi}{r\omega} + \frac{1}{r\pi}\left[\frac{\sin r\omega t}{r\omega}\right]_{\pi/\omega}^{2\pi/\omega} = \frac{\pi(\cos r\pi - 2)}{r\omega^2} \quad ...(7)$$

Substituting the solves of integrals from (5), (6) and (7) in (4), we have

$$A_r = \frac{4a}{T^2}\left[-\frac{\pi\cos r\pi}{r\omega^2} + \frac{2\pi(\cos r\pi - 1)}{r\omega^2} - \frac{\pi(\cos r\pi - 2)}{r\omega^2}\right]$$

$$= \frac{4a\pi}{T^2r^2\omega^2}\left[-\cos r\pi + 2\cos r\pi - 2 - \cos r\pi + 2\right]$$

$$= 0 \quad ...(8)$$

$$B_r = \frac{2}{T}\int_0^T y\cos r\omega t\ dt = \frac{2}{T}\left[\int_0^{T/2}\frac{2at}{T}\cos r\omega t\ dt + \int_{T/2}^{T}\frac{2a(T-t)}{T}\cos r\omega t\ dt\right]$$

$$= \frac{4a}{T^2}\left[\int_0^{T/2} t\cos r\omega t\ dt + \int_{T/2}^{T} T\cos r\omega t\ dt - \int_{T/2}^{T} t\cos r\omega t\ dt\right] \quad ...(9)$$

Now, let us solve the integrals

$$I_1 = \int_0^{T/2=\pi/\omega} t\cos r\omega t\ dt = \left[-t\frac{\sin r\omega t}{r\omega}\right]_0^{\pi/\omega} - \int_0^{\pi/\omega} -\frac{\sin r\omega t}{r\omega}dt$$

$$= \left[\frac{\cos r\omega t}{(r\omega)^2}\right]_0^{\pi/\omega} = \frac{1}{r^2\omega^2}(\cos r\pi - 1) \quad ...(10)$$

$$I_2 = T\int_{T/2}^{T}\cos r\omega t\ dt = T\left[-\frac{\sin r\omega t}{r\omega}\right]_{\pi/\omega}^{2\pi/\omega} = 0 \quad ...(11)$$

$$I_3 = \int_{T/2}^{T} t\cos r\omega t\ dt = \left[t\frac{\sin r\omega t}{r\omega}\right]_{\pi/\omega}^{2\pi/\omega} - \int_{\pi/\omega}^{2\pi/\omega}\frac{\sin r\omega t}{r\omega}dt$$

$$= 0 + \left[\frac{\cos r\omega t}{(r\omega)^2}\right]_{\pi/\omega}^{2\pi/\omega} = \frac{1}{(r\omega)^2}(1 - \cos r\pi) \quad ...(12)$$

Hence, $$B_r = \frac{4a}{T^2}\left[\frac{1}{r^2\omega^2}(\cos r\pi - 1) + 0\frac{1}{r^2\omega^2}(1 - \cos r\pi)\right]$$

$$= \frac{4a}{(2\pi/\omega)^2}\cdot\frac{2}{r^2\omega^2}(\cos r\pi - 1) = \frac{2a}{\pi^2r^2}(\cos r\pi - 1) \quad ...(13)$$

When r is even i.e., r = 2, 4, 6,..., cos $r\pi$ = 1.

$\therefore$ $$B_r = \frac{2a}{\pi^2r^2}(1–1) = 0 \quad ...(14)$$

When r is odd i.e., r = 1, 3, 5, ..., cos $r\pi$ = –1. Then

$$B_r = \frac{2a}{\pi^2 r^2}(-1-1) = -\frac{4a}{\pi^2 r^2} \qquad ...(15)$$

Substituting for A_0, A_r and B_r from (3), (8), (14) and (15) in (2), we get the Fourier series as

$$y = f(\omega t) = \frac{a}{2} - \frac{4a}{\pi^2 \times 1^2}\cos\omega t - \frac{4a}{\pi^2 \times 3^2}\cos 3\omega t - \frac{4a}{\pi^2 \times 5^2}\cos 5\omega t ..... - \frac{4a}{\pi^2 \times r^2}\cos r\omega t .....$$

or
$$y = \frac{a}{2} - \frac{4a}{\pi^2}\left[\cos\omega t + \frac{1}{3^2}\cos 3\omega t + \frac{1}{5^2}\cos 5\omega t + ... + \frac{1}{r^2}\cos r\omega t + ...\right] \qquad ...(16)$$

Thus the triangular wave has the axis $y = a/2$ above the time axis and the component terms are the odd harmonics with frequencies in the ratio 1 : 3 : 5 : ... . We may plot the different terms of the series graphically and find their resultant. First term represents the value of the coefficient $A = a/2$ and it gives the displacement of the axis of complex vibration curve from the time axis. 1st, 2nd and 3rd terms in the bracket are represented by dotted curves in **fig. 11**. Their resultant is obtained by the superposition of these three curves with constant $A = a/2$ and is shown by thick full line curve **(fig. 11)**. If a very large number of terms are considered, the resultant will resemble the triangular shape closely **(fig. 10)**.

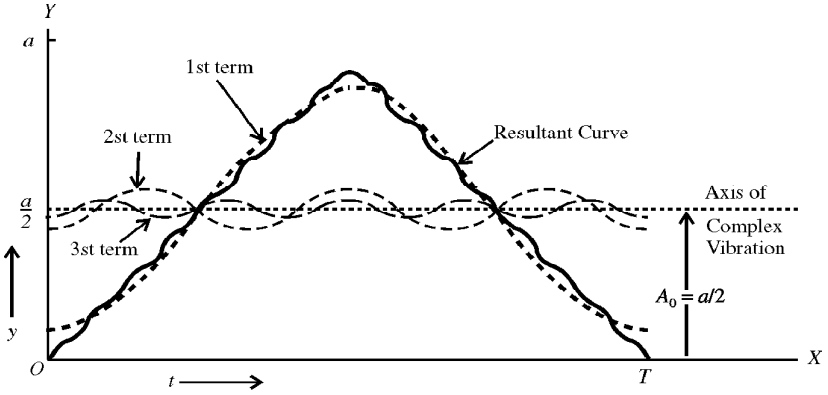


Fig. 11: Resultant of first three cosine terms of the series and mean displecement (a/2) of the axis of complex vibration.

5. Importance of Fourier Theorem

This theorem plays an important role in the analysis of complex periodic vibrations into harmonic components or in the synthesis of such vibrations from harmonic components. Fourier theorem has its applications to audio mechanical and electrical systems where periodic phenomena may occur. Stringed instruments, like piano, guitar, sitar and violin find important role in the world of music. These inslruments produce musical sound by transverse vibrations of stretched strings. These vibrations are in general complex and are analysed into harmonic complonents by applying Fourier's theorem. The harmonic component having the lowest frequency is called fundamental and the other components, differing in frequency to that of fundamental, are called overtones or harmonics. Fourier theory allows us to excite or stop particular harmonic components of the vibrating string. The quality of musical sound depends upon the presence of the number of harmonics and sometimes on the absence of particular harmonic. For example, in piano, the strings are struck approximately (1/7)th of the length so that the seventh harmonic which tends to dissonance with other notes in a chord is eliminated. Fourier theorem is also useful in the analysis of output of rectifier circuits and hence in the designing of suitable filter circuits.

Ex. 1. *Find the Founier series expansion for the function* $f(x) = 1$ *for* $-\pi \le x \le 0$ *and* $f(x) = 2$ *for* $0 \le x \le \pi$. **(O.U. 2014, 09)**

Sol. Let the Founier series be

$$f(x) = A_0 + \sum_{r=1}^{\infty} A_r \sin rx + \sum_{r=1}^{\infty} B_r \cos rx$$

Now,
$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{0} 1\times dx + \frac{1}{2\pi}\int_{0}^{\pi} 2\times dx$$

$$= \frac{1}{2\pi}[x]_{-\pi}^{0} + \frac{1}{2\pi}[2x]_{0}^{\pi} = \frac{1}{2\pi}\times\pi + \frac{1}{2\pi}\times 2\pi = \frac{1}{2}+1 = \frac{3}{2}$$

$$A_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin rx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\sin rx\,dx + \int_{0}^{\pi}\sin rx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos rx}{r}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-2\cos rx}{r}\right]_{0}^{\pi} = \frac{1}{\pi r}[-1+\cos r\pi] + \frac{1}{\pi r}[-2\cos r\pi + 2]$$

$$= \frac{1}{\pi r}(1-\cos r\pi)$$

When r is even, $A_r = 0$

When r is odd, $A_r = \frac{2}{\pi r}$

$$B_r = \frac{1}{\pi r}\int_{-\pi}^{\pi} f(x)\cos rx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos rx\,dx + \int_{0}^{\pi} 2\cos rx\,dx\right]$$

$$= \frac{1}{\pi r}\left[\frac{\sin rx}{r}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{\sin rx}{r}\right]_{0}^{\pi} = 0$$

Hence Fournier series is

$$f(x) = \frac{3}{2} + \sum_{r=1}^{\infty}\frac{2}{\pi r}\sin rx, \quad \text{when } r = 1,3,5....$$

or
$$f(x) = \frac{3}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ... + \frac{1}{r}\sin rx + ...\right]$$

Ex. 2. *Find the Fourier series exansion of* $f(x) = x$, $-\pi < x < \pi$. **(O.U. 2014, 09)**

Sol. Let the Founier series expansion of the given function be given by

$$f(x) = A + \sum_{r=1}^{\infty} A_r \sin rx + \sum_{r=1}^{\infty} B_r \cos rx$$

In order to have the expansion of Fourier series, we find the coefficients A_0, A_r and B_r.

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] = 0$$

$$A_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin rx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin rx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos rx}{r}\right) - 1\left(\frac{-\sin rx}{r^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos rx}{r}\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\pi\left(\frac{-\cos r\pi}{r}\right) - \pi\frac{\cos(-r\pi)}{r}\right] = \frac{-2\cos r\pi}{r}$$

$$B_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos rx\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin rx}{r}\right) + 1\left(\frac{\cos rx}{r^2}\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\cos rx}{r^2}\right]_{-\pi}^{\pi} = 0$$

Thus $$f(x) = \sum_{r=1}^{\infty}\left(\frac{-2}{r}\cos r\pi\right)\sin rx = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - .....\right]$$

Ex. 3. *Find the Fourier series componenets of a complex harmonic motion defined by*

$y = a$ when $0 < t < T/2$

$y = 0$ when $T/2 < t < T$. **(A.U. 2014, 11)**

Sol. The complex harmonic motion is represented in **fig.**

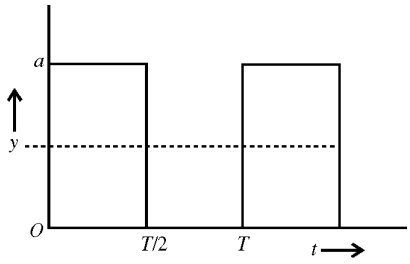


Fig. 12.

Let the Fourier series expansion of the function be

$$f(x) = A_0 + \sum_{r=1}^{\infty} A_r \sin r\omega t + \sum_{r=1}^{\infty} B_r \cos r\ \omega t$$

where $\omega = 2\pi/T$,

Now, $$A_0 = \frac{1}{T}\int_0^{T/2} a dt = \frac{a}{2}$$

$$A_r = \frac{2}{T}\int_0^{T/2} a\sin\left(\frac{2\pi rt}{T}\right) dt = \frac{2a}{T}\left[-\frac{T}{2\pi r}\cos\left(\frac{2\pi r}{T}\right)\right]_0^{T/2} = \frac{a}{\pi r}\left[1-\cos\pi r\right]$$

$$B_r = \frac{2}{T}\int_0^{T/2} a\cos\left(\frac{2\pi rt}{T}\right)dt = \frac{2a}{T}\left[\frac{T}{2\pi r}\sin\left(\frac{2\pi rt}{T}\right)\right]_0^{T/2} = 0$$

$\therefore$ $$f(x) = \frac{a}{2} + \sum_{r=1}^{\infty}\frac{a}{\pi r}[1-\cos\pi r]\sin r\omega t$$

or $$f(x) = \frac{a}{2} + \sum_{r=1}^{\infty}\frac{2a}{\pi r}\sin r\omega t\ ,\ \text{when } r \text{ is odd}$$

or $$f(x) = \frac{a}{2} + \frac{2a}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\cos\omega t + ...\right]$$

QUESTIONS

Long Answer Questions

1. State and explain Fourier's theorem. Describe how the Fourier coefficients can be evaluated. **(S.K.U. 2007; Krishna U. 2014)**

2. State Fourier's theorem. What are its limitations? Evaluate Fourier coefficients. **(A.N.U. 2012, T.U. 2014)**

3. State the Fourier theorem and evaluate the Fourier coefficients. **(K.U. 2005; A.N.U. 2005)**
4. State and explain Fourier theorem. Explain how you analyse periodic functions using Fourier theorem?
5. State and prove Fourier's theorem. Analyse a square wave with the help of Fourier Theroem. **(K.U. 2013; S.K.D.U. 2013)**
6. Discuss the application of Fourier's series for the analysis of a square wave. **(A.U. 2013; K.U. 2014; S.V.U. 2014)**
7. State Fourier's theorem. Obtain Fourier series for sqaure wave. **(S.K.U. 2005)**
8. Write down Fourier coefficients. Analyse saw-tooth waves. **(M.G.U. 2014, Dr. B.R.A.U. 2013)**
9. State the Fourier's theorem. Analyse a saw-tooth wave using Fourier's theorem. **(K.U. 2007)**
10. (a) What is Fourier theorem? Obtain the Fourier coefficients. **(O.U. 2008)**
(b) Analyse the triangular wave using Fourier theorem. **(O.U.2008)**
11. State and explain Fourier theorem. How do you use it to analyse a triangular wave. **(A.U. 2012, 07; S.K.U. 2014; S.U. 2014)**

Short Answer Questions

1. What is Fourier theorem?
2. Discuss the conditions for the application of Fourier theorem.
3. State Fourier theorem. What are its limitations? **(V.S.U. 2014)**
4. State and explain Fourier theorem. **(O.U. 2003; A.N.U. 2013; A.U. 2001)**
5. State and prove Fourier theorem. **(K.U. 2014)**
6. How do you evaluate Fourier coefficients? **(A.U. 2014)**
7. Analyse a square wave using Fourier theorem. **(Dr. B.R.A.U. 2014)**
8. Write the application and limitations of Fourier's theorem. **(A.N.U. 2014)**
9. State and explain Fourier theorem. Mention its applications. **(A.U. 2007)**
10. What are the limitations of Fourier theorem? **(O.U. 2013)**

PROBLEMS

1. Apply Fourier's theorem to analyse the square wave given below **(fig. 13)**.

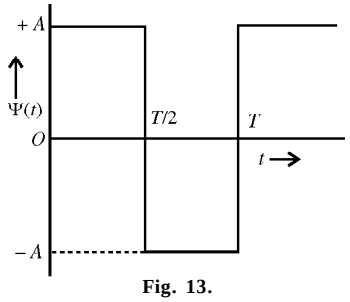


Fig. 13.

[Ans. $\Psi(t) = \frac{4A}{\pi}(\text{sing } \omega t + \frac{1}{3}\ 3\omega t + \frac{1}{5}\sin 5\omega t + ...)$**]**

[Hint. Mathematically the square wave may be expressed as:

$\Psi(t) = +A$ for $0 < t < T/2$
$\Psi(t) = -A$ for $T/2 < t < T$

2. Fig. 14 gives a periodic function. Obtain a Fourier analysis of this upto the fifth harmonic.

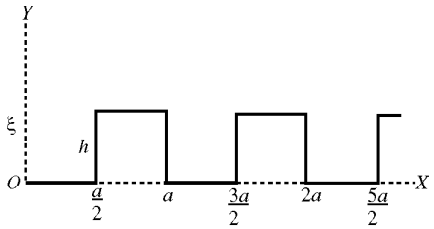


Fig. 14.

[Ans. $\xi = \frac{h}{2} - \frac{2h}{\pi}\left[\sin\frac{2\pi x}{a} + \frac{1}{3}\sin\frac{6\pi x}{a} + \frac{1}{5}\sin\frac{10\pi x}{a} + ...\right]$ **]**

[Hint. The given periodic function can be expressed as:

$$\xi = 0 \quad \text{for } 0 < x < a/2$$

$$\xi = 0 \quad \text{for } a/2 < x < a$$

3. Find the Fourier's series for the saw-tooth function given by

$$y = \frac{a}{T}t \quad \text{for } 0 < t < T$$

[Ans. $y = \frac{a}{2} - \frac{a}{\pi}\left(\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + ...\right)$ **]**

4. State the conditions under which Fourier's theorem may be applicable. Apply it to deduce the harmonic components of a saw-tooth wave of frequency n/2x and form given by

$$y = -\frac{a}{2} + a\frac{n}{2\pi}t \quad \text{for } t = 0 \text{ to } \frac{2\pi}{n}.$$

[Ans. $-\frac{a}{\pi}\left(\sin nt + \frac{1}{2}\sin 2nt + \frac{1}{3}\sin 3nt + ...\right)$ **]**.

5. Find the Fourier series of the periodic function $f(t)$, the period of which is T and the form of which in the first period, i.e., $0 \leq t \leq T$ is given by $f(t) = t\,(T - t)$.

[Ans. $y = \frac{T^2}{6} - \frac{T^2}{\pi^2}\left(\cos\omega t + \frac{1}{2^2}\cos 2\omega t + \frac{1}{3^2}\cos 3\omega t + ... + \frac{1}{r^2}\cos r\omega t + ...\right)$ **]**

6. Find the Fourier coefficients for the function represented by the following equations:

$y = 0$ for $-\frac{T}{2} < t < \frac{T_1}{2}$, $y = a$ for $-\frac{T_1}{2} < t < +\frac{T_1}{2}$, and $y = 0$ for $\frac{T_1}{2} < t < \frac{T}{2}$.

[Ans. $A_0 = \frac{T_1}{T}a$, $A_r = 0$ and $B_r = \frac{2a}{r\pi}\left(\frac{r\pi T_1}{T}\right)$;

$$y = \frac{aT_1}{T} + \frac{2a}{\pi}\left[\sin\left(\frac{\pi T_1}{T}\right)\cos\omega t + \frac{1}{2}\sin\left(\frac{2\pi T_1}{T}\right)\cos 2\omega t + \frac{1}{3}\sin\left(\frac{3\pi T_1}{T}\right)\cos\omega t + ... + \frac{1}{r}\sin\left(\frac{r\pi T_1}{T}\right)\cos r\omega t + ...\right]$$

]

7. Find the Fourier series for the periodic function shown in **fig. 15** given by

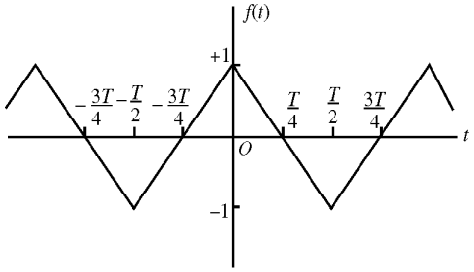


Fig. 15.

$$f(t) = \begin{cases} f(t) = 1 + \frac{4t}{T} & \text{for } -\frac{T}{2} < t < 0 \\ \text{and } f(t) = 1 - \frac{4t}{T} & \text{for } 0 < t < \frac{T}{2} \end{cases}$$

[Ans. $f(t) = \frac{8}{\pi^2}\left[\cos\omega t + \frac{1}{3^2}\cos(3\omega t) + \frac{1}{5^2}\cos(5\omega t) + ...\right]$ **]**

8. A saw-tooth wave is represented by **fig. 16**, which may be expressed as.

$y = f(t) = A\left(1-\frac{t}{2T}\right)$ from $t = 0$ to $t = T$

Find the fourier series for this saw-tooth wave.

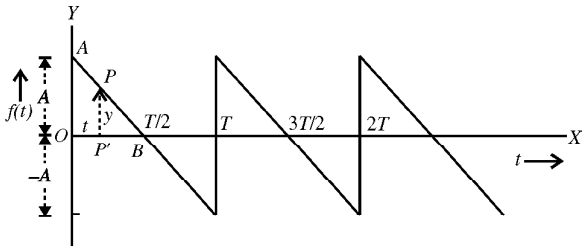


Fig. 16.

[Ans. $f(t) = \frac{2A}{\pi}\left(\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + ... + \frac{1}{r}\sin r\omega t + ...\right)$**]**

[Hint. $\frac{PP'}{0A} = \frac{P'B}{0B}$ or $\frac{y}{A} = \frac{(T/2)-t}{T/2}$ or $y = A\left(1-\frac{2t}{T}\right)$**]**

4

Vibrating Strings

1. Wave Motion: Transverse and Longitudinal Waves

The concept of waves is very fundamental in nature. The waves which move in the forward direction are called **progressive** or **travelling waves.** In a travelling wave, there is always a transmission of energy along the direction of propagation of the wave. However in stationary waves, there is no transference of energy from one place to another.

The waves which need material medium to reach from one place to another are called **mechanical waves.** For the propagation of such waves, the medium must have the property of elasticity and inertia. Sound waves, waves on a string etc. are examples of mechanical waves. The waves which can even travel also in vacuum are called **electromagnetic waves.** X-rays, light waves, radio waves etc. are examples of electro-magnetic waves.

Mechanical waves are of two types:

(1) Transverse Waves: The waves in which each particle of the medium executes oscillatory motion about its mean position in a direction perpendicular to the direction of wave propagation, are called **transverse waves**. For example, the waves produced by the transverse vibration of a stretched string.

(2) Longitudinal Waves: The waves in which each particle of the medium executes oscillatory motion about its mean position along the direction of wave propagation, are called **longitudinal waves**. Sound waves, produced in the air, and the waves transmitted along a stretched spring by compressing its one end then releasing it, are the examples of longitudinal waves.

2. Wave Pulse on a String and Equation of Wave

Consider a long string with one end fixed to a wall and the other end being held by a person so that the string is tight under tension. Let the person move his hand a little up and down producing a bump in the string near his hand **[fig. 1]**. This disturbance or bump travels along the string towards right with constant speed. **Fig. 1** also shows the state of the string at successive instants. In this process, the person does some work on the part of the string close to his hand, providing some energy to that part. This disturbed part exerts elastic force on the part of the string (with inertial mass) just right to it and transfers the energy. The bump thus moves towards right along the string. This process continues and the different parts of the string are successively disturbed, propagating the energy from left to right.

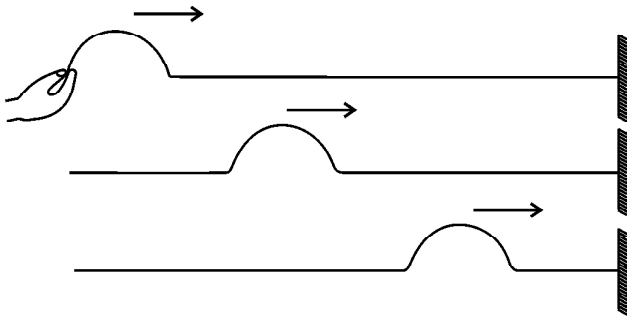


Fig. 1: Propagation of wave pulse on a string.

If the disturbance is localised only to a small part of space (string) at a time, it is said that a **wave pulse** is passing through that part of space. This wave pulse is formed, when the source (hand) is active only for a short time.

If the source, producing the disturbance, is active for relatively more time, repeating its motion several times, a **wave train** or a **wave packet** is obtained. For example, the person in **fig. 11·1** moves his hand up and down 5 times and then stops, a wave train with 5 loops will advance on the string towards right.

Equation of Travelling Wave

Consider a wave pulse or wave train, travelling on the string towards right with velocity v. If we take the source or the left end of the string as origin ($x = 0$) and X-axis along the string, the vertical displacement y of a particle at the origin is a function of time t (taking $t = 0$, when the particle is at the origin O in the beginning in the undisplaced position) i.e.,

$$y = f(t)$$

where $f(t)$ is a function of time, representing the displacement y of the particle at $x = 0$.

As the disturbance (or wave) is travelling towards right with speed v, this displacement y of the wave will reach a point P, distance x from O, after x/v time. Hence at any instant t the displacement y of the particle at P must be the same as the displacement at the origin x/v seconds earlier i.e.,

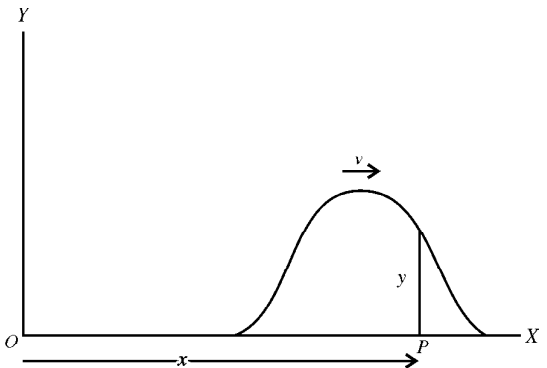


Fig. 2: Equation for a travelling wave : $y = f\left(t - \frac{x}{v}\right)$

$$y = f\left(t - \frac{x}{v}\right) \qquad ...(1)$$

This is the equation for wave, travelling along the positive direction of X-axis with constant velocity v **[fig. 2]**. The function f determines the shape and size of the wave. Such a wave is called a **travelling wave** or **progressive wave.** In case of a wave, the displacement y is a function of distance x and time t i.e., $y = y(x, t)$. It is to be noted that the time t and position x must appear in the wave equation in the combination $(t - x/v)$ only. For example,

$$y = a \text{ sins } \frac{2\pi}{T}\left(t - \frac{x}{v}\right),\ y = ae^{-\omega(t - x/v)}$$

etc. are valid wave equations.

If the wave travels in the negative X-direction, its equation will be represented as

$$y = f\left(t + \frac{x}{v}\right) \qquad ...(2)$$

The wave travelling in the positive X-direction may be represented as

$$y = f\left(\frac{vt - x}{v}\right) \text{ or } y = F(x - vt) \qquad ...(3)$$

where F is some other function of x and t.

If the source O vibrates in a *simple harmonic motion*, the equation for displacement y of the particle at the origin will be represented by $y = a \sin \omega t$ and the equation of the wave, produced by such a vibrating source, will be obtained to be

$$y = a \sin \omega(t - x/v) \qquad ...(4)$$

where a represents the amplitude and ω the angular frequency. If T is the time period of oscillation, then $T = 2\pi/\omega$ and frequency of oscillation $\nu = 1/T = \omega/2\pi$. The wave represented by eq. (4) is called a **sine wave** or **sinusoidal wave [fig. 3].**

Various alternative forms of equation of a travelling harmonic wave along positive *X*-axis can be written as

$$y = a \sin 2\pi\nu \left(t - \frac{x}{v}\right) \quad ...(4a)$$

or $$y = a \sin \frac{2\pi}{T}\left(t - \frac{x}{v}\right)$$

or $$y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) \quad ...(4b)$$

or $$y = a \sin \frac{2\pi v}{\lambda}\left(t - \frac{x}{v}\right)$$

or $$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad ...(4c)$$

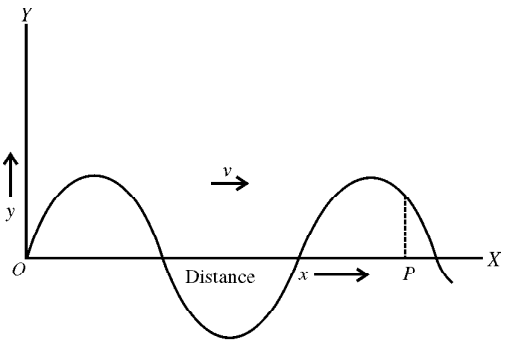


Fig. 3: Sine wave.

The quantity $2\pi/\lambda = 2\pi\,\bar{\nu} = k$ is called the **propagation constant,** where $\bar{\nu} = 1/\lambda$ is the wave number.

$$\therefore \quad y = a \sin (\omega t - kx) \quad ...(4d)$$

If ϕ is the phase constant, then

$$y = a \sin (\omega t - kx + \phi) \quad ...(4e)$$

Note that $$y = a \sin (kx - \omega t) \quad ...(4f)$$

is also a valid form of equation of a harmonic wave, travelling along positive *X*-axis, but it has a phase difference of π with the wave in eq. (4d).

Relation between Particle Velocity and Slope of Displacement Curve

The general equation of a wave, travelling along positive direction of *X*-axis with constant velocity *v* is given by

$$y = f\left(t - \frac{x}{v}\right) \quad ...(5)$$

where *y* is the displacement of a particle on the wave at any time *t* at a distance *x* from an arbitrary origin (or source).

Differentiating eq. (5) partially with respect to *t*, we get the expression for particle velocity i.e.,

$$\frac{\partial y}{\partial t} = f'\left(t - \frac{x}{v}\right) \quad ...(6)^*$$

Differentiating eq. (1) partially with respect to *x*, we get the expression for slope of the displacement curve i.e.,

$$\frac{\partial y}{\partial x} = -\frac{1}{v} f'\left(t - \frac{x}{v}\right) \quad ...(7)$$

From (6) and (7), we obtain

$$\frac{\partial y}{\partial x} = -\frac{1}{v}\frac{\partial y}{\partial t}$$

or $$\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x}$$

Particle velocity = – (**Wave velocity**) × (**Slope of displacement curve**)

---

* Let $t - x/v = u$. Then

$$y = f(u)$$

Now, $$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t} = f'(u)\frac{\partial (t - x/v)}{\partial t} = f'(u) = f'\left(t - \frac{x}{v}\right)$$

Also, $$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} = f'(u)\cdot\frac{\partial (t - x/v)}{\partial x} = f'(u)\left(-\frac{1}{v}\right) = -\frac{1}{v}f'\left(t - \frac{x}{v}\right)$$

3. Transverse Wave Propagation along a Stretched String: Velocity of Transverse Waves

In order to study the propagation of transverse waves along a string, first we must define a string. *By a string or wire in acoustics we mean a cord or wire whose length is large compared to its diameter, and which is perfectly uniform and flexible.* The string is stretched with a large tension so that the effects of gravitational forces are negligible.

Consider a string with one end fixed and the other end being pulled by a person with a tension *T*. Let the person produce a bump or wave pulse by moving his hand up and down. The wave pulse, so poroduced, travels along the string towards the fixed end. In such a case, the string is thrown into transverse vibration perpendicular to the direction of wave propagation.

Let us take *X*-axis along the direction of the undisplaced string and *Y*-axis in the direction of the displacement of the particles of the string at right angles to the *X*-axis **(fig. 4)**. Now, consider a small element *PQ* of the string of length equal to δx at a distance x from the origin. Let P' Q' be the displaced position of the element at any time *t*, with a displacement *y*. As the string is perfectly flexible, the tension will be the same at each point of the string along the tangent at that point. Hence in the displaced position of the string, the inclination will be different at different points of the string. If ϕ and $\phi + \delta\phi$, (where ϕ is itself very small) be the inclinations with *X*-axis of the tangents at P' and Q' points respectively, then resolving the tension acting at P' and Q' along *X* and *Y* axes, we have

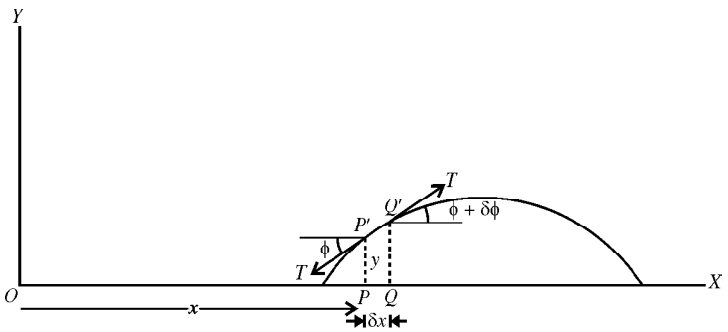


Fig. 4: Transverse wave propagation along a string.

$$\Sigma T_x = T\cos(\phi + \delta\phi) - T\cos\phi$$

$$\Sigma T_y = T\sin(\phi + \delta\phi) - T\sin\phi$$

As ϕ and $\delta\phi$ are small, hence

$$\cos(\phi + \delta\phi) \approx \cos\phi,\ \sin(\phi + \delta\phi) \approx \phi + \delta\phi \text{ and } \sin\phi = \phi$$

Thus, $\Sigma T_x = 0$ and $\Sigma T_y = T(\phi + \delta\phi) - T\phi = T\delta\phi$

For small angles, $\phi = \tan\phi$.

Hence, the resultant force along *Y*-axis is

$$\Sigma T_y = T\delta(\tan\phi) = T\delta\left(\frac{\partial y}{\partial x}\right) = T\,\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\delta x = T\,\frac{\partial^2 y}{\partial x^2}\delta x.$$

If *m* is the mass per unit length of the string, then the mass of small element δx will be $m\delta x$. If the acceleration of this element at the displacement *y* be $\partial^2 y/\partial t^2$, then according to Newton's second law of motion

$$m\delta x\frac{\partial^2 y}{\partial t^2} = t\frac{\partial^2 y}{\partial x^2}dx$$

or
$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\,\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

We know that the standard differential equation of the wave motion is given by

---

* If a string under tension is fixed at both ends and plucked at any point, a wave pulse travels toward either fixed end.

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(2)$$

where v is the velocity of the waves.

Hence eq. (1) represents the **differential equation for the transverse waves along a stretched string.** The velocity of the transverse waves along the string is obtained by comparing eqs. (1) and (2) i.e.,

$$v^2 = \frac{T}{m} \text{ or } v = \sqrt{\frac{T}{m}} \qquad ...(3)$$

Thus the velocity of transverse waves along a stretched string depends only on the applied tension T and mass per unit length (or linear mass density) m.

4. General Solution of the Wave Equation

The wave equation for transverse waves along a string is given by

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

where v represents the velocity of transverse waves along the stretched string.

Eq. (1) represents the one dimensional wave equation for transverse waves, travelling along X-axis (or string). Here y is the transverse displacement of a particle of the string and it is the function of x and time t. The general solution of eq. (1) must be a function of the linear combination of t and x i.e.,

$$y = f(t + \alpha x) \qquad ...(2)$$

where α is a constant.

Differentiating eq. (2) with respect to t, we obtain

$$\frac{\partial y}{\partial t} = f'(t + \alpha x)$$

Again differentiating with respect to t,

$$\frac{\partial^2 y}{\partial t^2} = f''(t + \alpha x) \qquad ...(3)$$

Differentiating eq. (3) with respect to x,

$$\frac{\partial y}{\partial x} = \alpha f'(t + \alpha x)$$

Again differentiating it with respect x,

$$\frac{\partial^2 y}{\partial x^2} = \alpha^2 f''(t + \alpha x) \qquad ...(4)$$

Substituting the values of $\frac{\partial^2 y}{\partial t^2}$ and $\frac{\partial^2 y}{\partial x^2}$ from eqs. (3) and (4) in eq. (1), we obtain

$$f''(t + \alpha x) = v^2 \alpha^2 f''(t + \alpha x)$$

whence
$$\alpha^2 = \frac{1}{v^2} \text{ or } \alpha = \pm \frac{1}{v} \qquad ...(5)$$

Hence the **general solution of the wave equation (1)** in view of eq. (2) is given by

$$y = f\left(t - \frac{x}{v}\right) + g\left(t + \frac{x}{v}\right). \qquad ...(6)$$

Significance of the General Solution of Wave Equation

When we look at the general solution of the wave equation, represented by eq. (6), we observe that the displacement of a particle at a point x is due to two waves: (i) one wave $[y_1 = f(t - x/v)]$ is travelling in the positive X-direction and (ii) the other wave $[y_2 = g(t + x/v)]$ is moving along the negative

X-direction. In case of a string, first wave, travelling along the positive *X*-direction, is the **incident wave** (say) and the other wave, moving along the negative *X*-direction is the **reflected wave.**

Simplest and most important case of the general solution of the wave equation is the **sinusoidal solution,** given by

$$y = a \sin \omega\left(t - \frac{x}{v}\right) + a \sin \omega \left[\left(t + \frac{x}{v}\right) + \phi\right] \quad ...(7)$$

where a is the amplitude of vibration, $\omega/2\pi$ (= ν) is the frequency of vibration and θ is the phase difference between the two waves. First sine wave [$y_1 = a \sin \omega (t - x/v)$] is the incident wave and the second sine wave $\left[y_2 = a\sin\left\{\omega\left(t + \frac{x}{v}\right) + \phi\right\}\right]$ is the reflected wave. By superposition, these two oppositely moving waves of the same amptitude, frequency and wavelength may produce **stationary waves**. This is the physical significance of the general solution of the wave equation.

5. Modes of Vibration of Stretched String Clamped (Fixed) at Both Ends (*Overtones and Harmonics*)

Consider a string of length l along X-axis. The string has been clamped (fixed) at $x = 0$ and $x = l$ **(fig. 5)**. Now, let a transverse sine wave of angular frequency ω (= $2\pi\nu$) and amplitude a given by

$$y_1 = a \sin \omega (t - x/v) \quad ...(1)$$

be travelling along the positive X-axis (string) wih speed v. This is the incident wave. The wave, reflected from the other end ($x = l$), moving in the negative X-axis, can be represented as

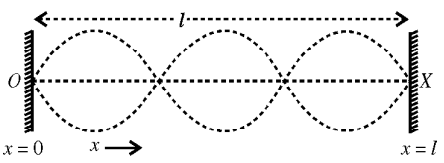


Fig. 5: Modes of vibration of stretched string damped at both ends.

$$y_2 = a \sin [\omega (t + x/v) + \phi] \quad ...(2)$$

where ϕ is the phase difference between the two waves.

The incident and the reflected waves superpose each other are identical in all respects but differ in phase. Hence they may give rise to standing waves. This is to be remembered that standing or stationary waves are formed by the superposition of two exactly similar waves travelling in opposite directions.

The resultant displacement of the particle of the string at the position x and time t is given by principle of superposition i.e.,

$$y = y_1 + y_2 = a \sin \omega \left(t - \frac{x}{v}\right) + a\sin\left[\omega\left(t + \frac{x}{v}\right) + \phi\right]$$

or,
$$y = 2a \cos \left(\frac{\omega x}{v} + \frac{\phi}{2}\right)\sin\left(\omega t + \frac{\phi}{2}\right)$$

Since $\omega/v = 2\pi\ \nu/v = 2\pi/\lambda = k$ is the wave number,

$$\therefore \quad y = 2a \cos \left(kx + \frac{\phi}{2}\right)\sin\left(\omega t + \frac{\phi}{2}\right) \quad ...(3)$$

where λ is the wavelength.

If standing waves are produced, the two ends of the string $x = 0$ and $x = l$ must be nodes because they are kept fixed. Thus, the **boundary conditions** are**:**

(i) $y = 0$ at $x = 0$ for all t

(ii) $y = 0$ at $x = l$ for all t

If we apply boundary condition (i) to eq. (3), we obtain

$$0 = \cos \phi/2 \text{ or } \phi/2 = \pi/2 \text{ or } \phi = \pi \quad ...(4)$$

Hence eq. (3) takes the form

$$y = -2a \sin kx \cos \omega t \quad ...(5)$$

For boundary condition (ii)

$$\sin kl = 0 \text{ or } kl = n\pi, \text{ where } n\ 1, 2, 3 \quad ...(6)$$

Hence $$l = \frac{n\pi}{k} = \frac{n\pi}{2\pi/\lambda} = \frac{n\lambda}{2} \quad ...(7)$$

Thus **in case of a string fixed at both ends standing waves are produced, when the length of the string is an integral multiple of $\lambda/2$.**

Harmonics and Overtones

From eq. (7)

$$\lambda = \frac{2l}{n} \quad ...(8)$$

We know that for a acoustic wave, $v = \nu\lambda$, where ν is the frequency of vibration. Hence

$$\nu = \frac{v}{\lambda} = \frac{vn}{2l} \quad ...(9)$$

If T is the tension in the string and m is the mass per unit length, then speed of the transverse waves v along the string is given by

$$\nu = \sqrt{\frac{T}{m}} \quad ...(10)$$

Thus, the expression for the frequency of vibration of the string is obtained to be

$$\nu = \frac{n}{2l}\sqrt{\frac{T}{m}} \quad ...(11)$$

where $n = 1, 2, 3.....$.

For $n = 1$, we obtain the lowest possible frequency, given by

$$\nu_1 = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad (\textit{fundamental tone or } 1\text{st } \textit{harmonic}) \; . \quad ...(12)$$

This is called the **fundamental frequency** of the string. Hence when the string is vibrating with fundamental frequency, the tone emitted by the string is known as **fundamental tone** or **first harmonic.**

The other natural frequencies of the string with which standing waves can be obtained for $n = 2$, 3, ... etc. are given by

$$\nu_2 = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad (1\text{st } \textit{overtone or } 2\text{nd } \textit{harmonic}) \quad ...(13)$$

$$\nu_3 = \frac{3}{2l}\sqrt{\frac{T}{m}} \quad (2\text{nd } \textit{overtone or } 3\text{rd } \textit{harmonic}) \quad ...(14)$$

etc. Thus, **possible allowed frequencies of vibration of a string clamped at both ends are an integral multiple of the fundamental frequency.** Instead of fundamental frequency the higher frequencies with which the string can vibrate are called **overtones.** In this way, $\nu_2 = 2\nu_1$ is the **first overtone** and $\nu_3 = 3\nu_1$ is the **second overtone** etc.

An integral multiple of the fundamental frequency is called its **harmonic.** In case of a string, clamped at boths ends, the frequencies of all overtones including fundamental are an integral multiple of the frequency of the fundamental i.e., in the ratio 1 : 2 : 3 etc. and hence they form a harmonic series. Thus **for a string fixed at both ends, all the overtones are harmonics of the fundamental frequency and all the harmonics of fundamental frequency are overtones.** Here, fundamental tone is the first harmonic, first overtone is the second harmonic, second overtone is the third harmonic etc.

Distinction between Harmonics and Overtones

The lowest natural frequency of an oscillating system is called its **fundamental tone.** The higher frequencies with which the system can vibrate are called **overtones.** In case of a string fixed at both ends, if ν_1 is the frequency of vibration of the fundamental, then $\nu_2 = 2\nu_1$, $\nu_3 = 3\nu_1$, etc. represent the overtones i.e., first overtone, second overtone, etc.

An integral multiple of the fundamental frequency is called its **harmonic.** In case of a string, clamped at both ends, fundamental tone is the first harmonic, first overtone is the second harmonic, second overtone is the third harmonic etc. There are cases where an overtone may have a frequency which is not exact multiple of the frequency of the fundamental. For example, when a bar, clamped at one end and free at the other, may vibrate transversely with higher frequencies or produce overtones whose frequencies are not an integral multiple of the fundamental. In this case, the overtones are not harmonics of the fundamental frequency because their frequencies (including fundamental) are in the ratio 1 : 6.27 : 17.55 etc.

Normal Modes of Vibration of the String

When all the particles of an oscillating system vibrate sinusoidally with the same frequency, the system is said to vibrate in a normal mode. If the string vibrates according to eq. (5) with some natural frequency, it is said to vibrate in a normal mode. Writing $k = n\pi/l$ from eq. (6) and $\omega = 2\pi\nu = n\pi v/l$ from eq. (9), eq. (5) can be written as

$$y = -2a \sin \frac{n\pi x}{l} \cos \frac{n\pi \nu t}{l} \qquad ...(15)$$

This equation represents the equation for the displacement of a particle of the string at the position x and time t in the nth **normal mode** or nth **mode**. The frequencies, given by $\nu = nv/2l$ (for n = 1, 2, 3...) of eq. (9) are called **normal frequencies** of vibration of the vibrating string fixed at both ends.

When n = 1, the string is said to vibrate in the **fundamental mode** or **first mode** with frequency $\nu_1 = v/2l$. In such a case, the equation of standing wave is

$$y = -2a \sin \frac{\pi x}{l} \cos \frac{\pi v t}{l} \qquad ...(16)$$

At any point x, the amplitude of vibration of a particle is $2a \sin (\pi x/l)$. The amplitude is zero at $x = 0$ and $x = l$. Its value is maximum at $x = l/2$ for $\sin \pi x/l = 1$. Thus in the fundamental mode of vibration of the string, fixed at both ends, we have two nodes (N) at the ends and one antinode (A) at the middle point of the string **[fig. 6 (a)]**.

When n = 2, the string vibrates in **second normal mode** with frequency $\nu_2 = v/l = 2\nu_1$. The equation of standing wave in this mode is

$$y = -2a \sin \frac{2\pi x}{l} \cos \frac{2\pi \nu \tau}{l} \qquad ...(17)$$

The amplitude $2a \sin 2\pi x/l$ is zero at $x = 0$, $l/2$ and l and is maximum ($2a$) at $x = l/4$ and $x = 3l/4$. Thus there are three nodes (N) at $x = 0$, $l/2$ and l and two antinodes (A) at $x = l/4$ and $3l/4$ **[fig. 6 (b)]**. In the second mode, the string will vibrate in two loops. Clearly, at any instant, all the particles of the first loop in between the nodes at $x = 0$ and $x = l/2$ will be in the same phase and simultaneously all the particles of the second loop in between the nodes at $x = l/2$ and $x = l$ will be in the same phase but opposite phase relative to the first loop. For example, the particle at $x = l/4$ will be in the opposite phase relative to the particle at $x = 3l/4$ because at $x = l/4$, $y = -2a \cos 2\pi vt/l$ and $x = 3l/4$, $y = +2a \cos (2\pi vt/l) = -2a \cos [(2\pi vt/l) + \pi]$.

In the **third mode** of vibration (n = 3), the frequency of the vibrating string is $\nu_3 = 3v/2l = 3\nu_1$ and the equation of standing wave is

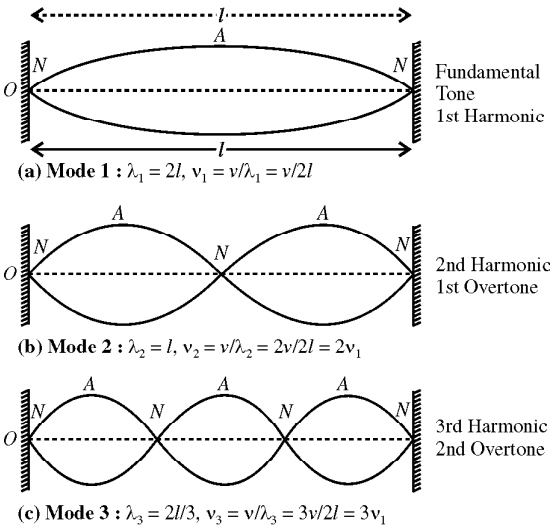


Fig. 6: First three normal modes of a string clamped at both ends.

$$y = -2a \sin\frac{3\pi x}{l} \cos\frac{3\pi vt}{l} \quad ....(17)$$

In this mode there will be four nodes at $x = 0$, $l/3$, $2l/3$ and l and three antinodes at $x = l/6$, $l/2$ and $5l/6$ **[fig. 6 (c)]**.

Thus if the string clamped at both ends is vibrating in the nth mode of vibration, the frequency of vibration is n times that of the fundamental ($\nu_n = n\nu_1$) and there will be $(n + 1)$ nodes and n antinodes.

When the string of a musical instrument such as sitar is pulled aside (plucked) at some point and released, its shape in vibration may not correspond to any normal mode, discussed above. Actually, the shape of the string is a combination of several normal modes and hence a combination of frequencies are emitted.

6. Plucked String

A stretched string is said to be excited by **plucking,** if it is pulled aside at any point through a small distance and then released.

We know that a string fixed at both ends, is capable of vibrating in all possible modes. When the string is vibrating in the nth mode, the equation for displacement is given by

$$y_n = A_n \sin\frac{n\pi x}{l} \cos\frac{n\pi vt}{l} \quad ...(1)$$

where l is the length of the string, A_n the maximum amplitude of the string in the nth mode and v the speed of waves.

The general vibration of the string, fixed at both ends, is that resulting from the superposition of all the normal modes i.e.,

$$y = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \cos\frac{n\pi vt}{l} \quad ...(2)$$

Suppose that the string of length l be pulled aside at a point distant a from the end $x = 0$ through a small transverse distance b and then released. The string is said to be *plucked* at $x = a$ and the resulting vibration is that due to *plucking.* The distance b is so small that the tension T in the string is assumed constant. The initial displacement at any point x of the string is given by **[fig. 7]**,

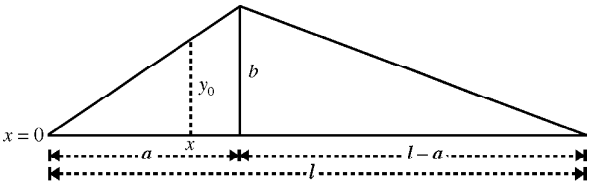


Fig. 7: Plucking of a string.

$$y_0 = \frac{b}{a}x \quad \left[\because \frac{y_0}{x} = \frac{b}{a}\right] \text{ from } x = 0 \text{ to } x = a \quad ...(3)$$

and

$$y_0 = \frac{b}{l-a}(l - x) \left[\because \frac{y_0}{l-x} = \frac{b}{l-a}\right] \text{ from } x = a \text{ to } x = l.$$

Multiply both sides of eq. (1) by $\sin\frac{n\pi x}{l}$ and integrate between the limits $x = 0$ to $x = l$ using proper values of y_0 in each range from (3). Then taking $t = 0$, we have

$$\int_0^a \frac{bx}{a} \sin\frac{n\pi x}{l}\,dx + \int_a^l \frac{b(l-x)}{(l-a)} \sin\frac{n\pi x}{l}\,dx = \int_0^l A_n \sin^2\frac{n\pi x}{l}\,dx$$

whence

$$A_n = \frac{2bl^2}{n^2\pi^2 a(l-a)} \sin\frac{n\pi a}{l} \quad ...(4)$$

Hence the general solution for the plucked string is

$$y = \frac{2bl^2}{\pi^2 a(l-a)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi x}{l} \cos\frac{n\pi vt}{l} \quad ...(5)$$

Some modes (harmonics) may be absent in (5), because the amplitude of vibration A_n in (4) depends on the distance a of the point of plucking. The nth mode of vibration or nth harmonic will be absent if

$$\sin\frac{n\pi a}{l} = 0 \text{ or } \frac{n\pi a}{l} = m\pi \text{ or } a = \frac{l}{n}m \quad ...(6)$$

where $m = 1, 2, 3....$ etc.

If the string is plucked at the centre (middle point), then $a = \frac{l}{2}$ and from (6) $\frac{l}{2} = \frac{l}{n}\ m$ or $n = 2m$ or $n = 2, 4, 6...$. Hence **2nd, 4th, 6th, ... modes (harmonics) will be absent for a string plucked at the mid point.** All these harmonics require a node at $l = a/2$. In other words, if the string is plucked at the middle point, the string cannot vibrate with frequencies which are even multiples ($n = 2, 4, 6...$) of its fundamental frequency.

If $a = l/3$, $n = 3m$ or $n = 3, 6, 9...$. Hence if a string is plucked at a distance $a = l/3$ from the fixed end, then 3rd, 6th, 9th, ... harmonics will be absent and the remaining modes will be present. All these harmonics need a node at $a = l/3$. Thus, in general **for a plucked string the harmonics (modes), having a node at the point of plucking, are absent**. This is called **Young's law.**

From (4), we see that **the amplitude in the *n*th mode of vibration is inversely proportional to n^2. Therefore the amplitude of higher harmonics fall off rapidly.** The amplitude of the fundamental mode is four times that of the second mode, nine times that of the third mode etc. i.e., the amplitude of successive modes is falling off rapidly.

It is to be noted that the virbation of the string can be stopped by touching the point of plucking after the plucked string is set into vibration. As the harmonics which have a node at the point of plucking are not excited in accordance with Young's law and the rest of the harmonics which do not have a node at the point of plucking are stopped by damping due to touching, in consequence all vibrations of the string will stop.

When the string is plucked at a single point by a sharp object, the emitted note is very brilliant due to the presence of a large number of modes as in case of the instrument, mandoline. However for a plucked string all the modes having a node at the point of plucking will be absent. In case the string is plucked over a finite length by a round object like finger, a less number of modes are excited so that the sound emitted is softer. The quality of sound given out by a harp is inferior when compared to that given out by a mandoline because the former is plucked by fingers.

7. Laws of Transverse Vibrations of Strings

The fundamental frequency of vibration of a string of length l (fixed at both ends), having a linear mass density m and being stretched under tension T, is given by

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad ...(1)$$

We can deduce three laws which are called **laws of transverse vibrations of a string** from the above equation.

(1) Law of length: When T and m are constants, $\nu \propto \frac{1}{l}$ *i. e., the fundamental frequency of vibration of a stretched string is inversely proportional to the length of the string when tension and mass per unit length are kept constants.*

(2) Law of tension: When l and m are constants, $\nu \propto \sqrt{T}$ i.e., *the frequency of the fundamental mode of a string is directly proportional to the square root of tension where the length and mass per unit length are kept constants.*

(3) Law of mass: When l and T are constants, $\nu \propto \frac{1}{\sqrt{m}}$ i.e., *the frequency of the fundamental mode of a string is inversely proportional to the square root of mass per unit length of the string when the length and tension are kept constants.*

As mass per unit length $m = \pi r^2 \times 1 \times \rho = \pi (D^2/4)\rho = \pi D^2\rho/4$, the frequency of vibration of the string can be written as

$$\nu = \frac{1}{2l}\sqrt{\frac{4T}{\pi D^2 \rho}} \text{ or } \nu = \frac{1}{lD}\sqrt{\frac{T}{\pi\rho}} \quad ...(2)$$

where D is the diameter of the string and ρ the density of its material.

Hence we may write two more laws, deduced from the law of mass.

(a) Law of diameter: When l, T and ρ are constants, $\nu \propto \frac{1}{D}$ i.e., *the frequency of transverse vibration of a string is inversely proportional to the diameter of the string, when length, tension and density of the material of the string are kept consstants.*

(b) Law of density: When l, T and D are constants, $\nu \propto \frac{1}{\sqrt{\rho}}$ i.e., *the frequency of vibration of a stretched string is inversely proportional to the square root of density of the material of the string, when length, tension and diameter are kept constants.*

These laws were first discovered by M. F. Mersenne in 1636 and can be experimentally verified by a sonometer.

8. Energy of Vibrating String: Energy Density

When a progressive transverse wave moves along a string, the energy and the disturbance are being transported from one place to another. In a wave motion, the particles of the string vibrate about their equilibrium position and the vibrating string possesses both potential and kinetic energy.

Consider a sine wave travelling on a stretched string along positive X-axis. The mass per unit length of the string is m and T is the tension. Let the equation of the sine wave be

$$y = a \sin \omega (t - x/v) \quad ...(1)$$

Consider a small element of length dx and hence mass mdx at x on the string. Assume that the tension T remains the same even when this small element dx vibrates. As the element is very small, we can also assume that velocity and acceleration of its particles to be the same.

Particle velocity at x, $\frac{\partial y}{\partial t} = \omega a \cos \omega (t - x/v)$

Particle acceleration at x, $\frac{\partial^2 y}{\partial t^2} = -\omega^2 a \sin \omega (t - x/v) = -\omega^2 y$

Kinetic energy of the element at displacement y,

$$K = \frac{1}{2}(mdx)\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2} mdx\, \omega^2 a^2 \cos^2 (t - x/v) \quad ...(2)$$

Force acting on the element, $F_y = (mdx)\frac{\partial^2 y}{\partial t^2} = -mdx\, \omega^2 y$

This is the restoring force acting on the element at the displacement y. Hence the total amount of work done on the element in displacing 0 to y is

$$W = -\int_0^y F_y dy = \int_0^y mdx\, \omega^2\, y\, dy$$

$$= mdx\, \omega^2 \frac{y^2}{2} = \frac{1}{2} mdx\, \omega^2 a^2 \sin^2 \omega (t - x/v)$$

This becomes the potential energy of the element at the displacement y i.e.,

Potential energy of the element at the displacement y,

$$U = \frac{1}{2} mdx\, \omega^2 a^2 \sin^2 \omega (t - x/v) \quad ...(3)$$

Hence the total energy of the element dx at the displacement y,

$$dE = K + U = \frac{1}{2} m\, dx\, \omega^2 a^2 \cos^2 \omega (t - x/v) + \frac{1}{2} mdx\, \omega^2 a^2 \sin^2 \omega (t - x\, v)$$

or $$dE = \frac{1}{2} mdx\, \omega^2 a^2 \quad ...(4)$$

The total energy per unit length of the string is called **energy density** (E) of the vibrating string. Hence the energy density is given by

$$E = \frac{dE}{dx} = \frac{1}{2} m\, \omega^2 a^2 = 2\pi^2 \nu^2 a^2 m \quad ...(5)$$

where $\omega = 2\pi\nu$ and ν is the frequency.

The **total energy per wavelength** $E(\lambda)$ in the string will be obtained by integrating the expression (4) from $x = 0$ to $x = \lambda$ i.e.,

$$E(\lambda) = \int_0^{\lambda} dE = \int_0^{\lambda} \frac{1}{2} m\omega^2 a^2 da = \frac{1}{2} \omega^2 a^2 m\lambda = 2\pi^2 \nu^2 a^2 m\lambda \quad ...(6)$$

Problem: *Show that for a travelling harmonic wave on a stretched string, average kinetic energy density is equal to average potential energy density when the average is taken over a wavelength. Show that the sum of two energy densities is equal to the total energy density.*

Solution: Equation of travelling harmonic wave on a string is

$$y = a \sin \omega \left(t - \frac{x}{v}\right) \quad ...(i)$$

Potential energy of an element of length dx of the string,

$$dU = \frac{1}{2} m\omega^2 a^2 \sin^2 \omega \left(t - \frac{x}{v}\right) dx \quad ...(ii)$$

Kinetic energy of an element of length dx of the string,

$$dK = \frac{1}{2} m\omega^2 a^2 \cos^2 \omega \left(t - \frac{x}{v}\right) dx \quad ...(iii)$$

Potential energy density at x, $U = \frac{dU}{dx} = m\omega^2 a^2 \sin^2 \omega \left(t - \frac{x}{v}\right)$ ...(iv)

Kinetic energy density at x, $K = \frac{dK}{dx} = \frac{1}{2} m\omega^2 a^2 \cos^2 \omega \left(t - \frac{x}{v}\right)$ ...(v)

Average potential energy density, when average is taken over a wavelength λ,

$$U_{av} = \frac{1}{\lambda} \int_0^{\lambda} U\, dx = \frac{1}{\lambda} \int_0^{\lambda} \frac{1}{2} m^2\omega^2 \sin^2\omega \left(t - \frac{x}{v}\right) dx$$

$$= \frac{1}{4} \frac{m\omega^2 a^2}{\lambda} \int_0^{\lambda} \left[1 - \cos 2\omega\left(t - \frac{x}{v}\right)\right] dx \qquad [\because \cos 2\theta = 1 - 2\sin^2\theta]$$

$$= \frac{m\omega^2 a^2}{4\lambda} \cdot \lambda = \frac{m\omega^2 a^2}{4} = \pi^2 \nu^2 a^2 m \quad ...(vi)$$

Similarly average kinetic energy density

$$K_{av} = \frac{1}{\lambda} \int_0^{\lambda} \frac{1}{2} m\omega^2 a^2 \cos^2 \omega \left(t - \frac{x}{v}\right) dx = \pi^2 \nu^2 a^2 m \quad ...(vii)$$

Thus from (vi) and (vii), $K_{av} = U_{av}$

Sum of $K_{av} + U_{av} = \pi^2 \nu^2 a^2 m + \pi^2 \nu^2 a^2 m = 2\pi^2 \nu^2 a^2 m$ = total energy density.

9. Energy Transport and Rate of Energy Flow

When a travelling wave is moving along a string, energy is being transported along the direction of propagation of the wave. Let a sine wave be moving with speed v along X-axis on a string stretched

with tension T. The equation for displacement in the sine wave is given by

$$y = a \sin \omega (t - x/v) \quad ...(1)$$

where a is the amplitude and $\omega = 2\pi\nu$ is the angular frequency.

Consider a portion of the string at any time t to the right of a point at x. The string on the left of the point exerts a force T (tension) on this part. The direction of this force is along the tangent to the string at x. If the tangent is making angle ϕ with the X-axis **[fig. 8],** then the component of the force along Y-axis is

$$F = -T \sin \phi$$

But here ϕ is small so that $\sin \phi = \phi = \tan \phi = \partial y/\partial x$.

Therefore, $$F = -T \frac{\partial y}{\partial x}.$$

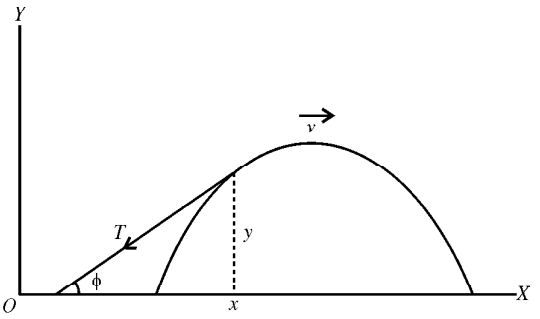


Fig. 8: Energy transport along a travelling wave on a string.

The point at x moves a distance δy in δt time. In this duration, the amount of energy given by the string on the left of the point to the string on the right will be equal to the work done by the force F i.e.,

$$\delta E = F\delta y = F \frac{\partial y}{\partial t} \delta t$$

$$= \left(-T \frac{\partial y}{\partial x}\right) \frac{\partial y}{\partial t} \delta t \quad ...(2)$$

From eq. (1)

$$\frac{\partial y}{\partial x} = -a\frac{\omega}{v} \cos \omega \left(t - \frac{x}{v}\right) \text{ and } \frac{\partial y}{\partial t} = a\omega \cos \omega \left(t - \frac{x}{v}\right)$$

$$\therefore \quad \delta E = \frac{Ta^2\omega^2}{v} \cos^2 \omega \left(t - \frac{x}{v}\right) \delta t$$

Hence the energy transmitted from left to right at x in one periodic time τ is given by

$$E(\tau) = \int_0^\tau \frac{\omega^2 a^2 T}{v} \cos^2 \omega \left(t - \frac{x}{v}\right) dt$$

$$= \frac{\omega^2 a^2 T}{v} \int_0^\tau \frac{1}{2}\left[1 + \cos 2\omega \left(t - \frac{x}{v}\right)\right] dt \quad [\because \cos 2\theta = 2\cos^2\theta - 1]$$

$$= \frac{\omega^2 a^2 T}{v} \cdot \frac{\tau}{2} \quad ...(3)$$

This expression gives *the energy transferred through a point of the string in a periodic time*.

But $\omega = 2\pi\nu$, ν is the frequency of the wave and $T = mv^2$ $[\because v = \sqrt{T/m}]$. Hence

$$E(\tau) = 2\pi^2\nu^2 a^2 mv\tau \quad ...(4)$$

But $\tau = 1/\nu$ and $v\,\tau = v/\nu = \lambda$, the wavelength. Hence

$$E(\tau) = 2\pi^2\nu^2 a^2\, m\lambda = E(\lambda) \quad ...(5)$$

This is the same expression as we obtained for the total energy per wavelength $E(\lambda)$ of the string. Thus, *the energy being transported in a periodic time through a point of the string is contained in a wavelength λ of the string.*

The average rate of energy or power being transmitted through the point x of the string is obtained from (4) as

$$P_{av} = \frac{E(\tau)}{\tau} = 2\pi^2\nu^2 a^2 mv \quad ...(6)$$

Thus, *the average rate of energy or power being transported across a point of the string in a travelling wave is proportional to the square of the frequency (ν) and square of the amplitude (a).*

We know that the energy per unit length of the string or the energy density is $E = 2\pi^2\nu^2a^2m$. Hence the average power being transported across a point of the string can be expressed as

$$P_{av} = E \times v \qquad ...(7)$$

i.e., **Average power transported = Energy per unit length of the string × Wave velocity.**

This is the energy contained in a length v of the string and passes through a point of the string per unit time. Hence expression (6) or (7) gives the rate of flow of energy i.e.,

$$\text{Rate of flow of energy} = 2\pi^2\nu^2a^2mv \qquad ...(8)$$

This quantity is called **energy current.**

10. Transverse Impedance

When a progressive wave is travelling along a string, the string presents an impedance to the propagation of the wave. *The ratio of instantaneous transverse force on any particle of the string to the instantaneous transverse velocity of that particle is called transverse or characteristics impedance of the string and it is denoted by Z.* Thus,

$$Z = \frac{\text{Transverse Force}}{\text{Transverse Velocity}} = \frac{F}{u} \qquad ...(1)$$

where transverse particle velocity $u = \partial y/\partial t$.

It is in analogy with the electrical impedance

$$Z = \frac{\text{Alternating e.m.f.}}{\text{Current}} = \frac{E}{i} \qquad ...(2)$$

Consider a transverse sine wave travelling on a stretched string along X-axis. The equation for displacement (y) of a particle of the string at an instant t is given by

$$y = a \sin \omega (t - x/v) \qquad ...(3)$$

where a is the amplitude and ω the angular frequency. The speed of the wave (v), when it is moving on a string, is represented by

$$v = \sqrt{\frac{T}{m}} \qquad ...(4)$$

where T is the tension in the string and m is the linear density or mass per unit length of the string.

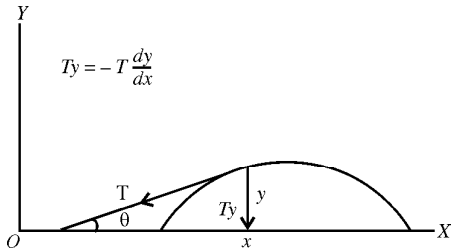


Fig. 9.

Let the tangent at a point x on the string make an angle ϕ with the X-axis at the particle displacement y **[fig. 9]**. Then the transverse force (F) acting on the particle at x is

$$F = -T \sin \phi = -T \tan \phi = -T \frac{\partial y}{\partial x} \qquad ...(5)$$

because for small ϕ, $\sin \phi = \phi = \tan \phi = \partial y/\partial x$.

From eq. (3)

$$\frac{\partial y}{\partial x} = -\frac{\omega a}{v} \cos \omega (t - x/v)$$

$$\therefore \quad F = \frac{\omega aT}{v} \cos \omega (t - x/v) \qquad ...(6)$$

Using eq. (3), particle velocity (u) is obtained to be

$$u = \frac{\partial y}{\partial t} = \omega a \cos \omega (t - x/v) \qquad ...(7)$$

Dividing eq. (6) by eq. (7), we obtain the expression for transverse impedance (Z) i.e.,

$$Z = \frac{F}{u} = \frac{T}{v} = \frac{mv^2}{v} = mv \qquad ...(8)$$

because from (4), $T = mv^2$.

Thus, *the transverse impedance offered by a string to travelling transverse waves is equal to the product of linear density (m) of the string and the velocity (v) of the waves.*

The transverse or characteristic impedance $Z = mv$ of the string can also be expressed as

$$Z = mv = m\sqrt{\frac{T}{m}} = \sqrt{mT}$$

or $$Z = mv = \frac{T}{v^2} \cdot v = \frac{T}{v} \qquad [\because v^2 = T/m \text{ or } m = T/v^2]$$

Thus $$Z = mv = \sqrt{mT} = \frac{T}{v} \qquad ...(9)$$

11. Reflection and Transmission of wave at the Boundary of Two Different Strings under Tension

At the junction $x = 0$ of the two strings, the boundary conditions are the following:

(i) At the boundary, the displacement is continuous *i.e.*,

$$y_i + y_r = y_t \qquad ...(1)$$

(ii) At the boundary, the transverse component of tension is continuous *i.e.*,

$$T_y^{\,i} + T_y^{\,r} = T_y^{\,t} \qquad ...(2)$$

From the second condition

$$Z_1 u_i - Z_1 u_r = Z_2 u_t \quad [\because T_y = Zu] \qquad ...(3)$$

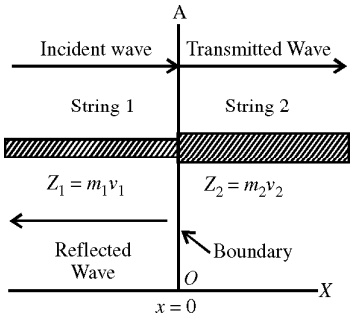


Fig. 10

Let the equation of waves be

$$\left.\begin{aligned} y_i &= a_i \sin\omega\left(t - \frac{x}{v_1}\right) \\ y_r &= a_r \sin\omega\left(t + \frac{x}{v_1}\right) \\ y_t &= a_t \sin\omega\left(t - \frac{x}{v_2}\right) \end{aligned}\right\} \qquad ...(4)$$

Hence, at $x = 0$ from eq. (1)

$$a_i + a_r = a_t \qquad ...(5)$$

and at $x = 0$ from eq. (3) $\left(u = \frac{\partial y}{\partial t}\right)$

$$Z_1 a_i - Z_1 a_r = Z_2 a_i$$

or $$a_i - a_r = \frac{Z_2}{Z_1} a_i \qquad ...(6)$$

From eqs. (5) and (6)

$$a_r = \frac{Z_1 - Z_2}{Z_1 + Z_2} a_i \text{ and } a_t = \frac{2Z_1}{Z_1 + Z_2} a_i \qquad ...(7)$$

The fraction of incident amplitude reflected is called **reflection coefficient of amplitude** (r) and that of incident amplitude transmitted is known at **transmission. coefficient of amplitude** (t). Thus

$$r = \frac{a_r}{a_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \quad ...(8)$$

and

$$t = \frac{a_t}{a_i} = \frac{2Z_1}{Z_1 + Z_2} \quad ...(9)$$

If we connet two similar strings, then $Z_1 = Z_2$ and $r = 0$ and $t = 1$; this means that there is no reflected wave and the amplitude a_t of transmitted wave is equal to the amplitude a_i of incident wave. This is called **impedance matching.** In the transportation of energy, impedance matching is very important. Long distance cables are matched in impedance at the joints so that there may be no loss of energy due to reflection.

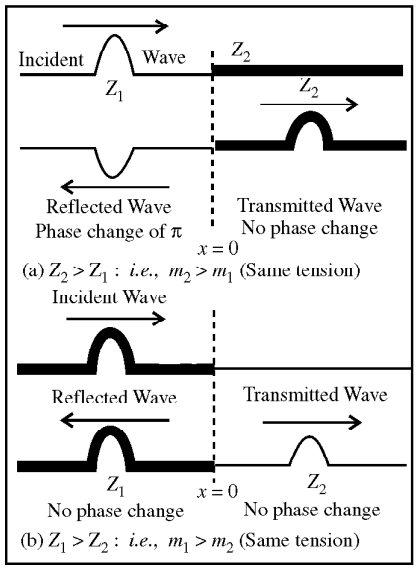


Fig. 11

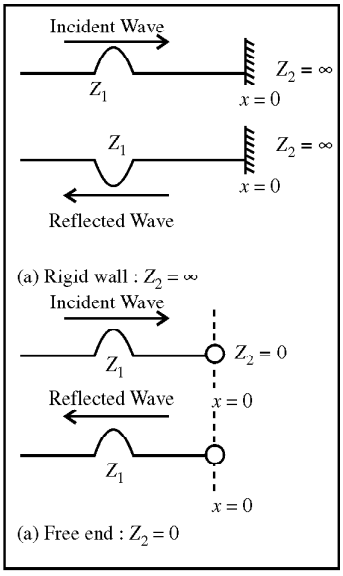


Fig. 12

If $Z_2 > Z_1$, r is negative. This means that the sign of reflected amplitude is opposite to that of incident amplitude. In other words, **there is a phase difference of π in reflection**. As t is always posituve, there is no phase difference in transmitted wave with respect to incident wave **[fig. 11(a)]**. As $Z_2 > Z_1$ or $m_2 > m_1$ ($Z = \sqrt{mT}$, T is the same in both strings), second string in heavies than first.

If $Z_1 > Z_2$, r or a_r/a_i is positive. Hence if a wave is reflected from the boundary of relatively less impedance, then **no phase difference occurs in the reflected wave due to reflection [fig. 11(b)]**. In this condition, $m_1 > m_2$ *i.e.*, first string is heavier than second.

If the second medium is a **rigid wall** instead a string, then $Z_2 = \infty$ so that $r = -1$ and $t = 0$. This means that

$$\frac{a_r}{a_i} = -1 \quad \text{or} \quad a_r = -a_i \text{ and } \frac{a_t}{a_i} = 0 \quad \text{or } a_t = 0$$

Thus an incident wave is totally reflected from a rigid wall with a phase difference of π **[fig. 12(a)]**. For example fixed end of a string in an example of rigid boundary.

If the end of the string is tree *i.e.*, $Z_1 = 0$, then

$$r = \frac{a_r}{a_i} = 1 \text{ and } t = \frac{a_t}{a_i} = 2$$

This means that the amplitude of reflected wave is equal to the amplitude of incident wave and no phase difference occurs.

Ex. 1. *A wave is travelling on a long stretched string along its length taken as positive X-axis. The equation of the wave is*

$$y = ae^{-\left(\frac{t}{T}-\frac{x}{\lambda}\right)}$$

where $a = 8$ *mm,* $T = 1$ *sec and* $\lambda = 4$ *cm.*

Find the velocity of the wave. In which direction is the wave travelling?

Sol. The equation of the travelling wave is

$$y = ae^{-\left(\frac{t}{T}-\frac{x}{\lambda}\right)} \quad ...(i)$$

which can be written as

$$y = ae^{-\frac{1}{T}\left(t-\frac{xT}{\lambda}\right)} \text{ or } y = a^{-\frac{1}{T}\ \left(t-\frac{x}{\lambda/T}\right)} \quad ...(ii)$$

This is the form of the general equation of the wave i.e.,

$$y = f\left(t-\frac{x}{v}\right) \quad ...(iii)$$

Comparing (ii) and (iii), we get

$$v = \frac{\lambda}{T} = \frac{4 \text{ cm.}}{1 \text{ sec.}} = \mathbf{4 \times 10^{-2}\ m/sec.}$$

Obviously, this wave is travelling along positive *X*-axis.

Ex. 2. *Displacement of a wave is* $y = 0.29 \cos (3x - 2t)$ *in S.I. units. Find amplitude, frequency and time period.* **(S. K. U. 2007)**

Sol. Displacement $y = 0.29 \cos (3x - 2t)$ ...(i)

Standard equation of a wave, $y = a \sin [(\omega t - kx) + \phi]$ ...(ii)

Eq. (i) can be written as

$$y = 0.29 \cos (2t - 3x) = 0.29 \sin [(2t - 3x) + \pi/2]$$

$\therefore$ Amplitude $a = \mathbf{0.29\ m.}$

$$\text{Frequency, } \nu = \frac{\omega}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi} = \mathbf{0.31\ Hz.}$$

$$\text{Time period, } T = \frac{1}{\nu} = \pi = \mathbf{3.14\ sec.}$$

Ex. 3. *Standing waves are produced by the superposition of two waves,* $y_1 = 10 \sin (3\pi t - 4x)$ *cm and* $y_2 = 10 \sin (3\pi t + 4x)$ cm. *Find the amplitude of motion at* $x = 18$ cm. **(A.U. 1995)**

Sol. In the standing waves, the resultant amplitude y is obtained as

$$y = y_1 + y_2 = 10 \sin (3\pi t - 4x) + 10 \sin (3\pi t + 4x)$$

$$\left[\because \sin A + \sin B = \sin\frac{A+B}{2}\cos\frac{A-B}{2}\right]$$

$$= 10 [2 \sin 3\pi t \cos 4x]$$

$$= 20 \cos 4x \sin 3\pi t$$

Here, the amplitude of motion $A(x) = 20 \cos 4x$

Now at $x = 18$ cm, $A(x = 18 \text{ cm}) = 20 \cos (4 \times 18) = \mathbf{19.5\ cm.}$

Ex. 4. *A string vibrates according to the equation*

$$y = 8 \sin\left(\frac{\pi x}{3}\right) \cos(20\pi t)$$

where x, y are in cm *and t in* sec. *Find the distance between two successive nodes and the speed of* particle of the string at a position $x = 1.5$ cm. *when* $t = 2.25$ sec.

Sol. At nodes, $y = 0$, i.e., $\sin\left(\frac{\pi x}{3}\right) = 0$

or $\frac{\pi x}{3} = n\pi$, where $n = 0, 1, 2, 3$

whence $x = 3\ n = 0, 3, 6, 9....$ cm

$\therefore$ Distance between two successive nodes = 3 cm.

$$\text{Velocity of particle, } \frac{dy}{dt} = -8 \sin\left(\frac{\pi x}{3}\right) 20\pi \sin(20\pi t)$$

When $x = 1.5$ cm and $t = 2.25$ sec, then the speed of the particle is

$$\frac{dy}{dt} = -8 \sin\left(\frac{\pi \times 1.5}{3}\right) 20\pi \times \sin(45\pi)$$

$$= -8 \sin\left(\frac{\pi}{2}\right) \times 20\pi \times \sin(45\pi) = -8 \times 20\pi \times 0 = \mathbf{0}.$$

Ex. 5. *The velocity of transverse wave in a* 150 cm *long wire of mass* 5 gm *subjected to a tension of* 1200 N *will be how much*? **(A.M.U. 2010; M.G.U. 2014; V.S.U. 2014; K.U. 2007)**

Sol. Velocity of transverse wave, $v = \sqrt{\frac{T}{m}} = \sqrt{\frac{1200}{5\times10^{-3}/1.5}}$ $\left[\because m = \frac{5 \text{ gm}}{150 \text{ cm}} = \frac{5\times10^{-3} \text{ kg}}{1.5 \text{ m}}\right]$

$$= \sqrt{36\times10^4} = \mathbf{600\ m/sec}$$

Ex. 6. *A steel wire* 50 cm *long has mass of* 5 gm. *It is stretched with a tension of* 400 N. *Find the frequency of the wire in fundamental mode of vibration.* **(A.N.U. 2006; O.U. 2010; K.U. 2006)**

Sol. Frequency in fundamental mode, $\nu = \frac{1}{2l}\sqrt{\frac{T}{m}}$

Here, $l = 50$ cm $= 0.5$ m, $m = 5 \times 10^{-3}$ kg$/0.5 = 10^{-2}$ kg

$\therefore$ $v = \frac{1}{2\times0.5}\sqrt{\frac{400}{10^{-2}}} = \frac{1}{1}\sqrt{400\times10^2} = \mathbf{200\ Hz.}$

Ex. 7. *A steel wire of* 1 cm *diameter is under tension of* 5 kN. *Find the velocity of the vibration produced. The density of steel material is* 7.7×10^3 kg/m^3. **(A.N.U. 2012)**

Sol. Velocity of wave, $v = \sqrt{\frac{T}{m}}$

where T is the tension in the wire and m = mass of 1 m of wire

$$= \text{volume of 1m of wire} \times \text{density} = \pi r^2 \times 1 \times \rho = \pi r^2 \rho$$

$$= 3.14 \times (0.5 \times 10^{-2})^2 \times 7.7 \times 10^3 \text{ kg/m}$$

$\therefore$ $\nu = \sqrt{\frac{5000}{3.14\times(0.5\times10^{-2})^2 \times 7.7\times10^3}} = \mathbf{90.95\ m/sec.}$

Ex. 8. *A flexible string of length* 1 m *and mass* 1 gm *is stretched to a tension T. The string is found to vibrate in three segments at a frequency* 612 Hz. *Calculate the tension in the string.* **(A. U. 2002)**

Sol. As the string is vibrating in three segments, we have

$$\therefore \qquad \nu = \frac{n}{2l}\sqrt{\frac{T}{m}} = \frac{3}{2l}\sqrt{\frac{T}{m}}$$

Putting the given values, we obtain

$$612 = \frac{3}{2\times 1}\sqrt{\frac{T}{10^{-3}}} \text{ or } \left(\frac{612\times 2}{3}\right)^2 = \frac{T}{10^{-3}}$$

$$\therefore \qquad T = \left(\frac{612\times 2}{3}\right)^2 \times 10^{-3} = \mathbf{0.166\ N.}$$

Ex. 9. *The diameter of the iron wire is* 1.2 mm. *If the speed of the transverse wave in the wire be* 50 m/sec, *then what is the tension in the wire?* (ρ_{iron} = 7.7 × 10^3 kg/m^3) **(A.U. 2012; O.U. 2008; A.N.U. 2010, 06)**

Sol. Speed of transverse waves in a string is given by

$$v = \sqrt{\frac{T}{m}} \text{ or } T = mv^2$$

where T is the tension and m is the mass of the string per unit length. Now, if r is the radius of the wire, then

m = mass of unit length
= volume of unit length × density of the wire
= $\pi r^2 \times 1 \times \rho$ [ρ = density of the material of wire]

$$= 3.14 \times \left(\frac{1.2\times 10^{-3}}{2}\right)^2 \times 7.7 \times 10^3$$

$$= 8.7 \times 10^{-3} \text{ kg/m}$$

$$\therefore \qquad T = 8.7 \times 10^{-3} \times (50)^2 = \mathbf{21.75\ newton.}$$

Ex. 10. *Calculate the speed of transverse waves in a wire of* 1 mm^2 *cross-section under the tension produced by* 0.1 kg *weight (specific gravity of material of wire* = 9.81 gm/cm^3 *and* g = 9.81 m/sec^2). **(S.V.U. 2009; A.N.U. 2002)**

Sol. $$\nu = \sqrt{\frac{T}{m}}$$

Here, $T = Mg$ = 0.1 × 9.81 = 0.981N, area of cross-section πr^2 = 1 mm^2 = $(10^{-3}\text{m})^2$ = 10^{-6} m^2, $\therefore$ $m = \pi r^2 \rho = 10^{-6}$ m^2 × (9.81 × 1000 kg/m^3) = 9.81 × 10^{-3} kg/m^3.

Hence, $$\nu = \sqrt{\frac{0.981}{9.81\times 10^{-3}}} = \sqrt{10^2} = \mathbf{10\ m/sec.}$$

Ex. 11. *The fundamental frequency of vibration of a stretched string of length* 1 m *is* 256 Hz. *Find the frequency of the same string of half the original length under indentical conditions.* **(S.V.U. 2010, 06; O.U. 01)**

Sol. Fundamental frequency $$\nu = \frac{1}{2l}\sqrt{\frac{T}{m}} \qquad ...(i)$$

Let the frequency be ν' of the same string of half the length *i.e.*, $l/2$.
Therefore

$$\nu' = \frac{1}{2(l/2)}\sqrt{\frac{T}{m}} \text{ or } \nu' = \frac{1}{l}\sqrt{\frac{T}{m}} \qquad ...(ii)$$

Dividing (ii) by (i), we get

$$\frac{\nu'}{\nu} = \frac{2l}{l} = 2, \quad \therefore \ \nu' = 2\nu$$

Given $\nu = 256$ Hz, $\quad \therefore \ \nu' = 2 \times 256 =$ **512 Hz.**

Ex. 12. *The linear density of vibrating string is* 1.3×10^{-4} kg/m. *A transverse wave propagating along the length of the string is discribed by the equation* $y = 0.03 \sin (x + 30t)$, *where x and y are in meters and t is in seconds. Determine the tension in the string.*

Sol. Equation of the transverse wave is

$$y = 0.03 \sin (x + 30t) \qquad ...(i)$$

Standard sine wave equation is

$$y = a \sin \omega (t - x/v) \qquad ...(ii)$$

Eq. (i) can be written as

$$y = 0.03 \sin 30 (t + x/30) \qquad ...(iii)$$

Comparing (ii) and (iii), we get

$$v = -30 \text{ m/sec}$$

This wave is travelling along negative X-axis with speed $v = 30$ m/sec.

If T is the tension in the string, then speed of the wave is given by

$$v = \sqrt{\frac{T}{m}}, \text{ whence } T = mv^2$$

Here, $m = 1.3 \times 10^{-4}$ kg/m and $v = 30$ m/sec

$\therefore \quad T = 1.3 \times 10^{-4} \times 30^2 =$ **0.117 N.**

Ex. 13. *Two identical guitar strings are tuned to the same frequency* 300 Hz. *The tension of one of the strings is increased by* 2%. *How many beats per second will be heard when the two strings are sounded together?* **(A.U. 2003)**

Sol. Let the initial tension in the two identical strings be T. The tension of one of the strings is increased by 2%. Hence the new tension in that string will be

$$T' = T + \frac{2}{100} \times T = T + 0.02\ T = 1.02\ T$$

Frequency, $\quad \nu = \frac{1}{2l}\sqrt{\frac{T}{m}}$

Initial frequency of each string, $\quad 300 = \frac{1}{2l}\sqrt{\frac{T}{m}} \qquad ...(i)$

Let the changed frequency of one of the string with increased tension ($1.02T$) be ν'. Hence

$$\nu' = \frac{1}{2l}\sqrt{\frac{1.02T}{m}} \qquad ...(ii)$$

Dividing eq. (ii) by eq. (i), we get

$$\frac{\nu'}{300} = \sqrt{\frac{1.02T}{T}} = \sqrt{1.02}$$

whence $\quad \nu' = 300 \times \sqrt{1.02} = 303$ Hz

$\therefore \quad$ Number of beats $= \nu' \sim \nu = 303 - 300 =$ **3 per sec.**

Ex. 14. *The fundamental frequency of a sonometer wire increases by* 5 Hz, *if its tension is increased by* 21%. *How will the frequency be affected if its length is increased by* 25%? **(O.U. 2007)**

Sol. Fundamental frequency, $\quad \nu = \frac{1}{2l}\sqrt{\frac{T}{m}} \qquad ...(i)$

When the tension is increased by 21%, the new tension becomes $T' = T + 0.21T = 1.21T$

$$\therefore \quad \nu + 5 = \frac{1}{2l}\sqrt{\frac{1.21T}{m}} \qquad ...(ii)$$

Dividing eq. (ii) by (i), we obtain

$$\frac{\nu+5}{\nu} = \sqrt{1.21} = 1.1 \text{ or } \nu + 5 = 1.1\nu \text{ or } 0.1\nu = 5$$

$$\therefore \quad \nu = 50 \text{ Hz.}$$

When the length is increased by 25%, the new frequency ν' is given by

$$\nu' = \frac{1}{2\times 1.25l}\sqrt{\frac{T}{m}} = \frac{\nu}{1.25} = \frac{50}{1.25} = \textbf{40 Hz.}$$

Ex. 15. *A flexible string of length* 1 m *and mass* 1 g *is stretched by a tension T. The string is found to vibrate in three segments at a frequency of* 512 Hz. *Calculate the tension.***(A. U. 2010; S.V.U. 2009)**

Sol. The frequency of the string, when it vibrates in three segments, is given by

$$\nu_3 = \frac{3}{2l}\sqrt{\frac{T}{m}}$$

Given $l = 1m$, $m = 10^{-3}$ kg/m and $\nu_3 = 512$ Hz.

$$\therefore \quad 512 = \frac{3}{2\times 1}\sqrt{\left(\frac{T}{10^{-3}}\right)} \text{ or } 512\times\frac{2}{3} = \sqrt{\left(\frac{T}{10^{-3}}\right)} \text{ or } \left(\frac{512\times 2}{3}\right)^2 = \frac{T}{10^{-3}}$$

$$\therefore \quad T = \left(\frac{512\times 2}{3}\right)^2 \times 10^{-3} = \textbf{116.5 N}$$

Ex. 16. *A string of length* 0.5 m *and linear density* 0.0001 kg/m *is kept under tension of* 1 N. *Find the frequency of the third overtone when it is plucked at its mid point.* **(K.U. 2014; A.N.U. 2007)**

Sol. frequency string Fundamental

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{m}} = \frac{1}{2\times 0.5}\sqrt{\frac{1}{0.001}} = 100 \text{ Hz.}$$

When the string is plucked at mid point, even harmonics of frequencies 2ν, 4ν, 6ν ... are absent. Hence the string vibrates with the frequencies ν, 3ν, 5ν, 7ν, ... Hence the frequency of the first, second and third overtones are 3ν, 5ν, 7ν or 300 Hz, 500 Hz, 700 Hz respectively.

Ex. 17. *A string of length* 2.6 m *and mass* 0.001 kg *kept under a tension* 1 N. *Find the fundamental frequency of the string. If the string is plucked at* 0.5 m *from one end, what frequencies would be stopped ?* **(S.V.U. 2007)**

Sol. The fundamental frequency of the string

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{m}}$$

Here, $T = 1$ N, length $l = 2.5$ m and $m = \frac{0.001}{2.5}$ kg/m

$$\therefore \quad \nu = \frac{1}{2\times 2.5}\sqrt{\frac{2.5}{0.001}} = \frac{1}{5}\sqrt{2500} = \frac{50}{5} = 10 \text{ Hz}$$

If the string is plucked at 0.5 m from one end, the point of plucking will be 0.5/2.5 = 1/5 of the length of the string (i.e., $a = l/5$). Thus harmonics of frequencies 5ν, 10ν, 15ν, ... i.e., **50 Hz, 100 Hz, 150 Hz**, ... would be absent or stopped.

Ex. 18. *A metallic wire of density* 8×10^3 kg/m^3 *is stretched between two clamps* 1 m *apart and is subjected to an extension of* 4.9×10^{-4} m. *What will be the lowest frequency of transverse vibration in the wire?* [*Young's modulus of material is* 8×10^{10} N/m^2]

Sol. The wire will have lowest frequency in the fundamental mode of vibration i.e.,

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{m}} = \frac{1}{2l}\sqrt{\frac{T}{A\rho}} \qquad [\because m = \pi r^2\rho = A\rho]$$

where A is the area of cross section and ρ is its density.

Young's modulus, $Y = \dfrac{\text{stress}}{\text{strain}} = \dfrac{T/A}{\Delta l/l}$, $\therefore\ T = YA\ \dfrac{\Delta l}{l}$

where l is the length and Δl is elongation.

In terms of Young's modulus, $\nu = \dfrac{1}{2l}\sqrt{\dfrac{YA\Delta l}{lA\rho}} = \dfrac{1}{2l}\sqrt{\dfrac{Y\Delta l}{l\rho}}$

Here, $l = 1$ m; $\Delta l = 4.9 \times 10^{-4}$ m; $Y = 8 \times 10^{10}$N/m^2 and $\rho = 8 \times 10^3$ kg/m^3

$$\therefore \quad \nu = \frac{1}{2\times 1}\sqrt{\frac{(8\times 10^{10})(4.9\times 10^{-4})}{1\times(8\times 10^3)}} = \mathbf{35\ Hz.}$$

Ex. 19. *When the length of the stretched wire on the sonometer is* 25 cm, *the note emitted by it is in unison with that of a tuning fork. When the length of the wire is changed to* 25.5 cm *without changing its tension,* 2 *beats per sec are heard on sounding the two together, what is the frequency of the fork*?

Sol. The fundamental frequency ν of the string of length l under tension T is given by

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{m}}$$

Since the length of wire is 25 cm = 0.25 m and it is vibrating in unision with a tuning fork. Hence the frequency of the fork is

$$\nu = \frac{1}{2\times 0.25}\sqrt{\frac{T}{m}} \qquad \text{...(i)}$$

Keeping the tension to be the same, when the length of the wire is changed to 25.5 cm = 0.255 m, the number of beats between the vibrating wire and the tuning fork are heard two. As the length of the wire is increasing, the frequency will decrease. Hence

$$\nu - 2 = \frac{1}{2\times 0{\cdot}255}\sqrt{\frac{T}{m}} \qquad \text{...(ii)}$$

Dividing equation (i) by (ii), we get

$$\frac{\nu}{\nu - 2} = \frac{0.255}{0.250} = \frac{51}{50}$$

or, $50\nu = 51\nu - 102$ or, $\nu = 102$

Thus, the frequency of the fork is **102.**

Ex. 20. *A wire of mass* 1 gm *and length* 75 cm *is making* 256 vib./sec. *under a tension supplied by a brass weight hanging vertically. On immersion the weight into water, the vibrating length of the wire is shortened by* 5 cm *to regain its original pitch. What is the density of brass?*

Sol. $n = \dfrac{1}{2l}\sqrt{\dfrac{T}{m}}$ or $T = 4l^2mn^2$

Here, $n = 256$ vib./sec, $l = 0.75$ m, $m = 10^{-3}/0.75 = 1/750$ kg/m

$\therefore \quad T = 4 \times 0.75 \times 0.75 \times (1/750) \times 256 \times 256$

$= \mathbf{196.61\ newton.}$

Let on immersion in water the tension be reduced to T'. To regain the same value of n, the length of wire $l = 75 - 5 = 70$ cm $= 0.7$ m.

$\therefore$ $T' = 5 \times (0.7)^2 \times (1/750) \times (256)^2 =$ **171 newton.**

$\therefore$Loss in weight due to immersion is $= (196.61 - 171) = 25.61$ newtons

$$\text{The relative density of brass} = \frac{\text{Wt. in air}}{\text{Loss in wt. due to immersion}}$$

$$= \frac{T}{T-T'} = \frac{196.1}{25.61} = 7.657$$

$\therefore$ Density of brass = **7.657 kg/m^3.**

Ex. 21. *Two wires of the same material and of the same cross-section are suspended on a sonometer. One is loaded with* 16 kg, *and the other with* 4 kg. *The first wire is tuned to the second harmonic of the second wire. If the second wire is* 1 m *in length, what is the length of the first wire*?

Sol. The fundamental frequency n_1 of the first wire is given by

$$n_1 = \frac{1}{2l_1}\sqrt{\frac{T_1}{m}} \qquad ...(i)$$

The two wires of are made of the same material and of same cross-section, therefore mass per unit length m of the two wires is the same. Now, the second harmonic n_2 of the second wire is

$$n_2 = \frac{2}{2l_2}\sqrt{\frac{T_2}{m}} \qquad ...(ii)$$

Since first wire is tuned to the second harmonic of the second wire,

$\therefore$ $n_1 = n_2$ i.e., $\frac{1}{2l_1}\sqrt{\frac{T_1}{m}} = \frac{2}{2l_2}\sqrt{\frac{T_2}{m}}$

Given: $T_1 = 16$ kg-wt, $T_2 = 4$ kg-wt, $l_1 = ?$, $l_2 = 100$ cm. $= 1$m.

$\therefore$ $\frac{2}{2\times 1}\sqrt{\frac{4}{m}} = \frac{1}{2l_1}\sqrt{\frac{16}{m}}$ or $l_1 =$ **1 m.**

Ex. 22. *The frequency of the fourth harmonic in a stretched string of length* 20 cm *is* 600 per sec. *What is the velocity of the wave in the string? If now the tension of the string is doubled, what will be the final velocity of the waves*?

Sol. Frequency of the 4th harmonic = 600 Hz

As it is four times the frequency of the fundamental,

$\therefore$ Frequency of the fundamental $= 600/4 = 150$ Hz

Velocity $\nu = \sqrt{T/m} = n\lambda = n \times 2l$

$= 150 \times 2 \times 0.2 = 60$ ms^{-1}

Finally tension is doubled, then

$\nu' = \sqrt{2T/m} = \sqrt{2} \times \sqrt{T/m} = \sqrt{2} \times 60 =$ **85 ms^{-1}.**

Ex. 23. *The speed of a transverse wave on a stretched string is* 500 m/s, *when it is stretched under a tension of* 19.6N. *If the tension is altered to a value of* 78.4 N, *what will be the speed of the wave ?* **(Krishna U. 2014, 08)**

Sol. Speed of a transverse wave in a string is given by

$$\nu = \sqrt{\frac{T}{m}}$$

If ν_1 and ν_2 are the speeds corresponding to tension in the string T_1 and T_2 respectively, then

$$\nu_1 = \sqrt{\frac{T}{m}} \text{ and } \nu_2 = \sqrt{\frac{T_2}{m}}$$

Hence $$\frac{\nu_2}{\nu_1} = \sqrt{\left(\frac{T_2}{T_1}\right)} \text{ or } \nu_2 = \nu_1\sqrt{\left(\frac{T_2}{T_1}\right)}$$

$$\nu_2 = 500 \times \sqrt{\left(\frac{78.4}{19.6}\right)} = 500 \times 2 = \mathbf{1000\ m/s}$$

Ex. 24. *A very long string of linear density* 0.01 kg/m *is stretched with a tension of* 400 N. *It is driven at one end in harmonic motion of amplitude* 2 cm *and frequency* 100 Hz. *Calculate the average rate of energy transported.*

Sol. *The average rate of energy transported is given by*

$\therefore$ $$P = 2\pi^2\nu^2a^2mv, \text{ where } v = \sqrt{T/m}$$

$$P = 2\pi^2\nu^2a^2\sqrt{mT}$$

Here, ν = 100 Hz, a = 2 cm = 0.02 m, m = 0.01 kg/m and T = 400 N

$\therefore$ $$P = 2 \times (3.14)^2 \times (100)^2 \times (0.02)^2\sqrt{0.01 \times 400} = \mathbf{157.75\ watt.}$$

Ex. 25. *The characteristic impedance of a string is* 3.2 kg/sec *and speed of the wave along the string is* 32 m/sec. *Find the tension in the stretched string and mass per unit length.*

Sol. Tension in the string, $T = Z\ v$

where Z is the impedance and v is the speed of transverse wave.

$\therefore$ $$T = 3.2 \times 32 = \mathbf{102.4\ N}$$

Linear density, $$m = \frac{Z}{v} = \frac{3.2}{32} = \mathbf{0.1\ kg/m.} \qquad [\because Z = mv]$$

Ex. 26. *Considering* $y = a_1 \sin(\omega t - kx) + a_2 \sin(\omega t + kx) + b_1 \cos(\omega t - kt) + b_2 \cos(\omega t + kx)$ *as the simple harmonic solution of the wave equation on a stretched string calmped rigidly at both the ends under tension T. Discuss the normal modes of vibration of the string.* **(K. U. 2013)**

Sol. $y = a_1 \sin(\omega t - kx) + a_2 \sin(\omega t + kx) + b_1 (\omega t - kx) + b_2 \cos(\omega t + kx)$ ...(i)

The stretched string is clamped rigidly at both ends. Hence the boundary conditions are

(1) $y = 0$ at $x = 0$ at all t

(2) $y = 0$ at $x = l$ at all t

Applying condition (1) in eq. (i), we obtain

$$0 = a, \sin \omega t + a_2 \sin \omega t + b_1 \cos \omega t + b_2 \cos \omega t$$

or $$0 = (a_1 + a_2) \sin \omega t + (b + b_2) \cos \omega t$$

$$A \sin \omega t = 0 \text{ and } \cos wt = 0, \text{ hence}$$

As $a_1 + a_2 \neq 0$ and $b_1 + b_2 \neq 0$,

$\therefore$ $$a_2 = -a, \text{ and } b_2 = -b_1$$

Therefore eq. (i) can be written as $y = a_1[\sin(\omega t - kx) - \sin(\omega t + kx)]$

$$+ b_1[\cos(\omega t - kx) - \cos(\omega t + kx)]$$

or $$y = 2a_1 \cos \omega t \sin kx + 2b_1 \sin \omega t \sin kx$$

or $$y = -2 \sin kx\ [a_1 \cos \omega t - b_1 \sin \omega t] \qquad ...(ii)$$

This represents a stationary wave formed by incident and reflected waves. Now, applying condition (2), we get

$$0 = -2 \sin kl\ [a_1 \cos \omega t - b_1 \sin \omega t]$$

As $$\cos \omega t \neq 0 \text{ and } \sin \omega t \neq 0,$$

$$\sin kl = 0 \text{ or } kl = n\pi \qquad ...(iii)$$

where $$n = 1, 2, 3 \ldots\ldots .$$

Hence $k = \frac{n\pi}{l}$ ...(iv)

But $k = \frac{2\pi}{\lambda}$, $\therefore$ $\frac{2\pi}{\lambda} = \frac{n\pi}{l}$ or $\lambda = \frac{2l}{n}$ ...(v)

For an acoustic wave $v = \nu\lambda$

$\therefore$ $\nu = \frac{v}{\lambda}$ or $\nu = \frac{vn}{2l}$ ...(vi)

If T is the tension in the string, $v = \sqrt{\frac{T}{m}}$ and hence

$$\nu_n = \frac{n}{2l}\sqrt{\frac{T}{m}} \quad \text{...(vii)}$$

For $n = 1$, the string is vibrating in fundamental mode with frequency given by

$$\nu_1 = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad \text{with } \lambda_1 = 2l \quad \text{[using eq. (v)]}$$

For $n = 2$ and $n = 3$ the string vibrates in second and third modes of vibration, with frequencies given by

$$\nu_2 = \frac{1}{l}\sqrt{\frac{T}{m}} \text{ with } \lambda_2 = l \text{ and } \nu_3 = \frac{3}{2l}\sqrt{\frac{T}{m}} \text{ with } \lambda_3 = \frac{2l}{3}.$$

In the *n*th mode of vibration

$$\nu_3 = \frac{n}{2l}\sqrt{\frac{T}{m}} \text{ with } \lambda_n = \frac{2l}{n}. \quad \text{...(viii)}$$

Ex. 27. *The strings of linear densities μ_1 and μ_2 are joined together and stretched with tension T. A transverse wave is incident on the boundary. Find the fraction of incident amplitude reflected and transmitted at the boundary if* $\frac{\mu_2}{\mu_1} = 4$ *and* $\frac{1}{4}$. **[S.V.U. 2010]**

Sol. Characteristic impedance of one string $Z_1 = \sqrt{\mu_1 T_1}$

Characteristic impedance of second string $Z_2 = \sqrt{\mu_2 T_2}$

$\therefore$ $\frac{Z_1}{Z_2} = \sqrt{\left(\frac{\mu_1}{\mu_2}\right)}$

Case (i): $\frac{\mu_2}{\mu_1} = 4$. *i.e.*, $\frac{Z_1}{Z_2} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

Hence the fraction of incidence amplitude reflected at the boundary is

$\therefore$ $\frac{a_r}{a_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{(Z_1/Z_2) - 1}{(Z_1/Z_2) + 1} = \frac{\frac{1}{2} - 1}{\frac{1}{2} + 1} = \frac{-\frac{1}{2}}{\frac{3}{2}} = -\mathbf{\frac{1}{3}}$

and the fraction of incidenct amplitude transmitted is

$$\frac{a_t}{a_i} = \frac{2Z_1}{Z_1 - Z_2} = \frac{2(Z_1/Z_2)}{(Z_1/Z_2) + 1} = \frac{2 \times \frac{1}{2}}{\frac{1}{2} + 2} = \mathbf{\frac{2}{3}}$$

Case (ii): $\frac{\mu_2}{\mu_1} = \frac{1}{4}$. In this case, $\frac{Z_1}{Z_2} = \sqrt{\frac{4}{1}} = 1$, Hence

$$\frac{a_r}{a_i} = \frac{2-1}{2+1} = \mathbf{\frac{1}{3}} \text{ and } \frac{a_t}{a_i} = \frac{2 \times 2}{2+1} = \mathbf{\frac{4}{3}}.$$

QUESTIONS

Long Answer Questions

1. Distinguish between transverse waves and longitudinal waves. Give examples. **(O.U. 2002)**

2. What are transverse waves? Derive the expression for the velocity of a transverse wave along a stretched string and discuss the solution of wave equation. **(A.N.U. 2014; K.U. 1997)**

3. Set-up the differential equation for a transverse wave on a stretched string. Obtain the frequencies of vibration for a string of length '*l*' rigidly supported at $x = 0$ and $x = 1$. **(O.U. 2007)**

4. What is a transverse wave? What is the differential form of the wave equation? Find out a solution for the wave equation.

5. Obtain the general solution of wave equation and mention its significance. **(A.U. 2007)**

6. Obtain general solution of wave equation on strings and derive an expression for energy density. **(N.U. 2001; S.K.U. 2001)**

7. Obtain an expression for velocity of transverse waves along stretched string. **(K.U. 2014; O.U. 2009; 08 A.N.U. 2007; S.V.U. 2014)**

8. What are transverse waves? Obtain an expression for the velocity of transverse wave in a stretched string. What are fundamental frequency and harmonics? **(S.U. 2014; Dr. B.U. 2014; K.U. 2005)**

9. Derive an expression for the speed of propagation of a transverse wave on a stretched string. How standing waves are formed in stretched strings? **(A.U. 2014, 2012)**

10. What are transverse waves? Mention the laws of transverse vibrations of strings. Obtain an expression for the frequency of bibration in a stretched string. **(O.U. 2014, 2012)**

11. State the laws of transverse vibrations of strings. Formulate the differential equation for a transverse motion in a stretched string. **(O.U. 2013)**

12. What is a transverse wave? Obtain the equation for the velocity of a transverse wave in a stretched string. Write the laws of transverse vibrations of the string. **(A.N.U. 2013; A.U. 2007, 03)**

13. Write the necessary theory giving corresponding diagrams that explain different modes of vibration of stretched string clamped at both the ends. **(O.U. 2005)**

14. Discuss the modes of vibration of a stretched string clamped at both ends? What are overtones? **(A.U. 2007, 05)**

15. Derive an expression for the displacement of a plucked string. What is the displacement corresponding to the first overtone when the string is plucked at the centre? **(A.N.U. 2006)**

16. Discuss briefly the mathematical theory of the modes of vibration of a plucked string. How does the quality of sound depend on the position at which a string under tension is plucked?

17. What are transverse waves? Obtain an equation for the frequency of vibration in a stretched string. Derive an expression for the energy of vibration. **(O.U. 2007; A.N.U. 2002)**

18. What are the transverse waves? Obtain an expression for the total energy per wavelength in a string when transverse wave travels on it. **(A.U. 1997)**

19. Define energy density in the wave motion along a string and obtain an expression for it. **(O.U. 1995)**

20. Obtain an expression for energy density of a wave travelling along a string. **(S.V.U. 2003; O.U. 2002)**

21. Derive an equation for the total energy per unit length in a string when a transverse wave travels in it. **(S.K.U. 2007)**

22. Discuss the energy transport in the strings and obtain an expression for energy desnity. **(K.U. 2007)**

23. Obtain the equation for the velocity of transverse waves in stretched string. Explain transverse impedance. **(Dr. B.R.A.U. 2013)**

24. Show that the characteristic impedance offered by a string to the travelling waves is $Z = \sqrt{\rho T}$, where ρ is the linear density and T is the tension with which it is stretched. **(O.U. 2008)**

Short Answer Questions

1. What is a pulse on a string?

2. We represent a transverse wave on a string by

$$y = a \sin (bx + ct),$$

where a, b and c are positive. In which direction, the wave is moving?

3. Write the differential form of the wave equation. What is its solution?

4. Obtain the relation between the particle velocity and the slope of the displacement curve. **(A. N.U. 2003)**

5. Explain the formation of stationary waves in a stretched string.

6. Write equations of motion of first three normal modes of vibration of a stretched string, fixed at both ends.

7. Show that a stretched string fixed between two points produces all possible frequencies. Explain how ? **(K.U. 1990)**

8. State the laws of transverse vibrations of stretched string. **(M.G.U. 2014; A.N.U. 2008; O.U. 2007, 01; S.V.U. 2007, 03; S.K.U. 2007)**

9. Explain the formation of harmonics and overtones. **(A.U. 2014, 07)**

10. Explain fundamental frequency, overtones and harmonics. **(A.U. 2012; A.N.U. 2008, 07)**

11. Distinguish between harmonics and overtones. **(O.U. 2014, 05; A.N.U. 2006)**

12. Explain the energy transport in vibrating strings. **(Krishna U. 2014; Dr. B.R.A.U. 2013; S.V.U. 2007, 05; A.U. 2005; A.N.U. 2012)**

13. Distinguish between wave velocity and group velocity of waves along strings. **(K.U. 2005)**

[**Hint.** Equation of a transverse wave along a string is

$$y = a \sin (\omega t - kx)$$

The relation between ω and k for transverse waves on a uniform string is

$$\omega = kv \quad \text{or} \quad \omega = k\sqrt{T/m}$$

$\therefore$ Wave velocity, $v_p = \dfrac{\omega}{k} = \sqrt{\dfrac{T}{m}}$ and Group velocity, $v_g = \dfrac{d\omega}{dk} = \sqrt{\dfrac{T}{m}}$

Hence $v_p = v_g = \sqrt{\dfrac{T}{m}}$.

Hence for transverse waves along a string the wave velocity is equal to group velocity and these waves are non-dispersive.]

14. Explain transverse impedance. **(T.U. 2014; K.U. 2014; A.U. 2007)**

15. Define energy density in the wave motion along a string and obtain an expression for it. **(R.U. 2014)**

16. Derive an expression for transverse impedance of a string. **(V.S.U. 2014)**

PROBLEMS

1. The equation of a wave travelling on a stretched string along the X-axis is given by

$$y = ae^{-\left(\frac{x}{\alpha}+\frac{t}{T}\right)}.$$

Find the speed of the wave and the direction in which the wave is travelling.

[**Ans.** α/T; along negative X-axis]

2. A travelling wave propagates according to the expression

$$y = 0.03 \sin (3x - 2t),$$

where y is the displacement at position x at time t. Taking the units to be in SI, determine (a) the amplitude, (b) the wavelength, (c) the frequency and (d) the period of the wave.

[**Ans.** (a) 0.03 cm; (b) 2.09 cm; (c) 0.31 Hz; (d) 3.14 sec]

3. Find the velocity of propagating of a transverse wave, the displacement of which is given by in S.I. system as $y = 2 \sin \pi (100t - x/60)$

[**Ans.** 3000 m/sec]

4. A string vibrates according to the equation $y = 5 \sin (\pi x/3) \cos (40\pi t)$, where x, y are in cm and t in sec. Find the distance between two successive nodes and the speed of particle of the string at a position $x = 1.5$ cm, when $t = 9/8$ sec and $9/16$ sec.
[**Ans.** 3 cm; 0, 628 cm/sec]

5. The equation for the vibration of a string, fixed at both ends, vibrating in its third harmonic, is given by

$$y = a \sin kx \cos \omega t$$

where $a = 0.5$ cm, $k = 0.314$ cm^{-1}, $\omega = 600\pi$ sec^{-1}.
Find the following:
(a) Frequency of vibration, (b) Position of nodes, (c) Length of the string, and (d) The wavelength and the speed of two travelling waves, producing the above vibration.
[**Ans.** (a) 300 Hz; (b) 0, 0.1 m, 0.2 m, 0.3 m; (c) 0.3 m; (d) 0.2 m, 60 m/sec]

[**Hints.** $\lambda = 2\pi/k$, $l = n\lambda/2$ and $v = \nu\lambda = \frac{\omega}{2\pi}\lambda$]

6. Standing waves are produced by superposition of two waves:

$$y_1 = 10 \sin (3\pi t - 4x) \text{ and } y_2 = 10 \sin (3\pi t + 4x).$$

Find the amplitude of motion at $x = 16$. **(K.U. 2005)**
[**Ans.** 7.06 length unit]
[**Hint.** $y = y_1 + y_2 = 10 \sin (3\pi t - 4x) + 10 \sin (3\pi t + 4x)$ or $y = 20 \cos 4x \sin 3\pi t$; Amplitude $A(x) = 20 \cos 4x$; Amplitude at $x = 16$, $A(x = 16) = 20 \cos 4 \times 16 = 20 \times 0.353 =$ **7.06**]

7. If the frequency of stretched string of length 1 m is 256 Hz, find the frequency of a string of half the length under identical condition. **(K.U. 2005; A.N.U. 2008, 07)**
[**Ans.** 512 Hz]

[**Hint.** $256 = \frac{1}{2l}\sqrt{\frac{T}{m}}$, $\nu = \frac{2}{2l}\sqrt{\frac{T}{m}}$ $\therefore \nu = 2 \times 256 =$ **512 Hz**]

8. A steel wire 50 cm long has mass of 5 gm. It is stretched with a tension of 400 N. Find the frequency of the wire in fundamental mode of vibration.
[**Ans.** 200 Hz]

9. A flexible string of length 1 m and mass 1 gm is stretched by a tension T. The string is found to vibrate in three segments of frequency of 512 Hz. Calculate the tension. **(A.N.U. 1995)**
[**Ans.** 116.5 N]

10. Calculate the speed of transverse waves in a wire of radius 0.05 cm under the tension produced by 1.0 kg weight. The density of the material of wire is 9.8×10^3 kg/m^3.
[**Ans.** 35·7 m/sec]
[**Hint.** 1 kg wt. = 1×9.8 N = 9.8 N; $m = \pi r^2 \rho$]

11. (a) Calculate the speed of the transverse waves in a wire of 1 mm^2 cross-section under the tension produced by 0.1 kg wt. (specific gravity of material of wire = 9.81 gm/cm^3 and $g = 9.81$ m/s^2) **(A.N.U. 2003, 02)**
[**Ans.** 10 m/sec]
(b) A wire of mass 0.001 kg and length 2.5 m is under tension of 1 N. Find the fundamental frequency of the wire. **(A.N.U. 2014)**
[**Ans.** 10 Hz]

12. A string of length 8 m fixed at both ends has a tension of 49 N and a mass of 0.04 kg. Find the speed of transverse waves on this string. **(A. U. 2003)**
[**Ans.** 98.99 m/sec]

13. Two strings vibrating side by side in a guitar have the same fundamental frequency of 256 Hz. If the tension of one of the strings is increased by 3%, find the number of beats heard. **(A.U. 2001)**
[**Ans.** 4 beats per sec]

14. The speed of a transverse wave on a stretched string is 500 m/sec when it is stretched under a tension of 19.6 N. If the tension is changed to a value of 78.4 N, find the speed of the wave. **(A.N.U. 2005)**
[**Ans.** 1000 m/sec]

[**Hint.** $v_1 = \sqrt{\frac{T_1}{m_1}}$ and $v_2 = \sqrt{\frac{T_2}{m_2}}$ $\therefore$ $\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}$]

15. A metallic wire of radius 1 cm has a length of 1 metre. It is fixed at both ends and is subjected to a tension of 10^4N. Calculate the fundamental frequency and the frequencies first and second overtones and the corresponding wavelength. (Density of copper is 8920 kg/m^3.)
[**Ans.** 298.7 Hz, 597.4 Hz, 896.1 Hz; 2 m, 1m, 0.67 m]

16. The fundamental frequency of a sonometer wire increases by 5 Hz, its tension is increased by 21%. How will the frequency be affected if its length is increased by 10%?
[**Ans.** Frequency = 45.45 Hz]

17. Two tuning forks produce 4 beats per sec when sounded together. One of them is in unison with a length of 120 cm. of a sonometer wire and other with 125 cm of the same wire under the same tension. Calculate the frequencies of the forks.
[**Ans.** 96 Hz and 100 Hz]

18. A fork and a string of length 100 cm give 4 beats per sec. The string is made shorter without any change of tension until it is in unison with the fork. If its length is now 99 cm, what is the frequency of the fork ?
[**Ans.** 400 Hz]

19. A metal wire of diameter 1 mm is held on two knife-edges separated by a distance of 0.5 m. The tension in the wire is 100 N. The wire, when vibrating with its fundamental frequency, produces 5 beats per second with a vibrating tuning fork. The tension in the wire is then reduced to 81 N. When the two are again excited, beats are heard again at the same rate. Calculate: (i) the frequency of the fork, (ii) the density of the material of the wire.
[**Ans.** (i) 95 second^{-1}; (ii) 12.7×10^3 kg/m^3]

[**Hint.** $(\nu + 5) = \frac{1}{2\times 0.5}\sqrt{\frac{100}{\pi(0.5\times10^{-3})^2\times\rho}}$

$(\nu - 5) = \frac{1}{2\times 0.5}\sqrt{\frac{81}{\pi(0{\cdot}5\times10^{-3})^2\times\rho}}$, where ν = frequency of fork]

20. The density of the material of the wire is 7.5×10^3 kg/m^3. If the stress on the wire be 3.0×10^8 N/m^2, then what will be the speed of the transverse wave in the wire?
[**Ans.** 200 m/s]

21. *P* and *Q* are two wires whose fundamental frequencies are 256 Hz and 382 Hz respectively. How many beats in two seconds will be heard by the third harmonic of *P* and the second harmonic of *Q*.
[**Ans.** 8]

22. A string of length 0.5 m and linear density 0.0001 kg m^{-1} kept under a tension 1 N. Find the first three overtones of the string when it is plucked at its mid-point.
[**Ans.** 300 Hz, 500 Hz, 700 Hz]

23. A string of mass 2 gm/m carries progressive waves of amplitude 1.5 cm, frequency 60 Hz and speed 200 m/sec. Calculate (a) the energy per metre length of the wire and (b) the rate of energy transport in the wave.
[**Ans.** (a) 0.032 J; (b) 0.64 watt]

24. A steel wire is stretched by a force of 100 N. If the mass per unit length of the wire is 2.5 gm/m, find the transverse impedance.
[**Ans.** 0.5 N m^{-1} - sec.]

●

5

Vibrations of Bars

1. Introduction

A bar or rod is considered to be a body of homogeneous and isotropic material whose length is large in comparison to its diameter. A bar may be made to vibrate *longitudinally, transversely* or *torsionally*. The exact mode of vibration depends both on the way of stroking and on the manner the bar is clampled. For the vibration of a bar, it is not needed to be stretched like a wire, because in place of tension, it is the elasticity of the material of the bar which restores its successive portions to their original relative positions.

2. Longitudinal Vibrations in a Solid Bar – Establishment of Wave Equation and Velocity of Longituidinal Waves

When a bar is stroked at one end with a hammer or rubbed along its length with a resined cloth, *longitudinal vibrations* are set up in it, moving along its length with a velocity depending upon the elasticity and density of the material of the bar. Cross-sections of the bar vibrate to and fro about their mean positions in the direction of the axis and the elements of the bar are extended or contracted as the longitudinal wave passes through them.

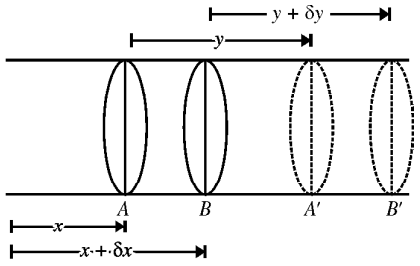


Fig. 1: Longitudinal vibrations in a bar.

Suppose that longitudinal waves are travelling along an elastic solid bar. Consider two close planes A and B, perpendicular to the axis of the bar at distances x and $x + \delta x$ respectively from some arbitrary origin [**fig. 1**]. If at any instant A' and B' are the positions of the planes A and B respectively, then the displacements at x and at $x + \delta x$ are y ($= AA'$) and $y + \delta y$ ($= BB'$) respectively. Hence the initial length AB between the two planes $= \delta x$.

Change in this length $= A'B' - AB = (\delta x + \delta y) - \delta x = \delta y$

$[\because \quad A'B' = (x + \delta x) + (y + \delta y) - (x + y) = \delta x + \delta y]$

$\therefore$ Longitudinal strain $= \delta y/\delta x = \partial y/\partial x$ ...(1)

The change in length of the element AB of the bar occurs due to the difference of force acting on the two planes. The force acting on the plane A is given by

F_a = stress × area of cross-section of the bar.

If α is the area of cross-section of the bar, the Young's modulus of its material is given by

$$Y = \frac{\text{stress}}{\text{strain}} \text{ or stress } = Y\frac{\partial y}{\partial x} \text{ or } \frac{F_a}{\alpha} = Y\frac{\partial y}{\partial x}$$

$$\therefore \qquad F_a = \alpha Y\frac{\partial y}{\partial x}$$

Similarly the force acting on the plane B is given by

$$F_b = F_a + \delta (F_a) = F_a + \frac{\partial}{\partial x}(F_a)\delta x$$

or, $$F_b = \alpha Y \frac{\partial y}{\partial x} + \alpha Y \frac{\partial^2 y}{\partial x^2} \delta x$$

$\therefore$ Resultant force on the element

$$AB = F_b - F_a = \alpha Y \frac{\partial^2 y}{\partial x^2} \delta x .$$

If ρ is the density of material of the bar, then the mass of the element $AB = \rho\alpha\delta x$. If the acceleration of this small element AB is $\partial^2 y/\partial t^2$, then according to Newton's second law of motion, we have

$$\rho \alpha \delta x \frac{\partial^2 y}{\partial t^2} = \alpha Y \delta x \frac{\partial^2 y}{\partial x^2}$$

or $$\frac{\partial^2 y}{\partial t^2} = \frac{Y}{\rho} \frac{\partial^2 y}{\partial x^2} \quad ...(2)$$

We know that the standard differential equation of the wave motion is given by

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad ...(3)$$

where v is the velocity of the waves.

Hence eq. (2) is the *differential equation for the longitudinal waves in a bar.* The velocity of the longitudinal waves in the bar is obtained by comparing eqs. (2) and (3) i.e.,

$$v^2 = \frac{Y}{\rho} \text{ or } v = \sqrt{\frac{Y}{\rho}} \quad ...(4)$$

Thus, the *velocity of longitudinal waves in a bar depends upon the density and Young's modulus of its material.*

3. Wave Equation for Longitudinal Vibrations and its General Solution

The wave equation for longitudinal vibrations in a bar is given by

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

where $v = \sqrt{\frac{Y}{\rho}}$. This is the one dimensional wave equation for longitudinal waves, travelling along X-axis (axis of the bar). Here y is the longitudinal displacement of a section of the bar and it is the function of x (distance from arbitrary origin) and time t. The general solution of eq. (5) must be a function of the linear combination of t and x i.e.,

$$y = f(t + \alpha x) \quad ...(2)$$

where α is a constant.

Differentiating eq. (2) with respect to t, we obtain

$$\frac{\partial y}{\partial t} = f'(t + \alpha x)$$

Again differentiating with respect to t,

$$\frac{\partial^2 y}{\partial t^2} = f''(t + \alpha x) \quad ...(3)$$

Differentiating eq. (2) with respect to x,

$$\frac{\partial y}{\partial x} = f'(t + \alpha x)$$

Again differentiating with respect to x,

$$\frac{\partial^2 y}{\partial x^2} = \alpha^2 f''(t + \alpha x) \quad ...(4)$$

Substituting the values of $\frac{\partial^2 y}{\partial t^2}$ and $\frac{\partial^2 y}{\partial x^2}$ from eqs. (3) and (4) in eq. (1), we obtain

$$f''(t + \alpha x) = v^2\alpha^2 f''(t + \alpha x)$$

Whence $$\alpha^2 = \frac{1}{v^2} \text{ or } \alpha = \pm \frac{1}{v} \quad ...(5)$$

Hence the general solution of the wave equation (1) in view of eq. (2) is given by

$$y = f\left(t - \frac{x}{v}\right) + g\left(t + \frac{x}{v}\right) \quad ...(10)$$

Clearly the displacement of a particle at a point x in the bar is due to two waves:

(i) One wave $[y_1 = f(t - x/v)]$ is travelling in the positive X-direction

and (ii) the other wave $[y_2 = g(t + x/v)]$ is moving along the negative X-direction.

In case of a solid rod, first wave, travelling along the positive X-direction, is the incident wave (say) and the other wave moving along the negative X-direction is the reflected wave.

Simplest and most important case of the general solution of the wave equation is the *sinusoidal solution,* given by

$$y = a \sin \omega\left(t - \frac{x}{v}\right) + a \sin \omega\left(t + \frac{x}{v} + \phi\right) \quad ...(11)$$

where a is the amplitude of vibration, $\omega/2\pi$ $(= \nu)$ is the frequency of vibration and ϕ is the phase difference between the two waves. First sine wave $[y_1 = a \sin \omega(t - x/v)]$ is the incident wave and the second sine wave $[y_2 = a \sin \{\omega(t + x/v) + \phi\}]$ is the reflected wave. By superposition, these two waves may produce *stationary waves.*

4. Stationary Longitudinal Waves in Bars

Consider a rod of length l and take O as the origin of X-axis. Let

$$y_1 = a \sin \omega\left(t - \frac{x}{v}\right) \quad ...(1)$$

represent a plane longitudinal sine wave travelling along the rod with amplitude a.

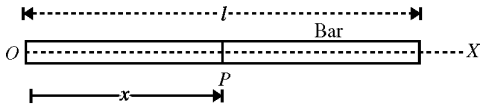


Fig. 2.

The wave reflected from the other end $(x = l)$ can be represented as

$$y_2 = a \sin\left[\omega\left(t + \frac{x}{v}\right) + \phi\right] \quad ...(2)$$

where ϕ is the phase difference.

The incident and reflected waves may give rise to stationary waves so that the displacement at a distance x from the origin is given by

$$y = y_1 + y_2$$

or, $$y = a \sin\omega\left(t - \frac{x}{v}\right) + a \sin\left[a \sin\omega\left(t + \frac{x}{v}\right) + \phi\right] \quad ...(3)$$

In fact, this is the general solution of the wave equation in the sinusoidal form. Thus the general solution may represent stationary waves produced in a bar. Eq. (3) can be written as

$$y = 2a \cos\left(\frac{\omega x}{v} + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right)$$

or, $$y = A \cos\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right) \quad ...(4)$$

where $A = 2a$, is the maximum value of the amplitude and $k = \omega/v = 2\pi/\lambda$ is the wave number corresponding to the wavelength λ.

5. Longitudinal Vibrations in a Bar, Free at Both Ends (Free-free Bar)

The general solution of the wave equation for the longitudinal vibrations in a bar is given by

$$y = A \cos\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right) \quad ...(1)$$

When the standing waves are formed, the free ends of the bar $x = 0$ and $x = l$ must be antinodes. Thus, we have the *boundary conditions* (in view of vanishing of $F = \alpha Y dy/dx$ at free ends)

(i) $$\frac{dy}{dx} = 0 \text{ at } x = 0 \text{ for all } t$$

and (ii) $$\frac{dy}{dx} = 0 \text{ at } x = l \text{ for all } t \quad ...(2)$$

Differentiating eq. (1) with respect to x, we get

$$\frac{dy}{dx} = -A\,k \sin\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right) \quad ...(3)$$

For boundary condition (i) i.e., at $x = 0$, $\frac{dy}{dx} = 0$, from eq. (3), we have

$$0 = \sin\frac{\phi}{2} \text{ or, } \phi = 0 \quad ...(4)$$

Hence, eq. (1) becomes

$$y = A \cos kx \sin \omega t \quad ...(5)$$

The second boundary condition (ii) will give by using eq. (3) (for $\phi = 0$)

$$\sin kl = 0 \text{ or, } kl = n\pi, \text{ where } n = 1, 2, 3,... \quad ...(6)$$

whence $$l = \frac{n\pi}{k} = \frac{n\pi}{2\pi/\lambda} \text{ or } l = \frac{n\lambda}{2} \quad ...(7)$$

Thus, *in case of a bar, free at both ends, standing waves are produced, when the length of the bar is an integral multiple of $\lambda/2$.*

We know that for acoustic waves,

$$v = \nu\lambda \text{ or } \nu = \frac{v}{\lambda} \quad ...(8)$$

where ν is the frequency of vibration.

From eq. (7) $$\lambda = \frac{2l}{n} \quad ...(9)$$

Hence, $$v = \frac{n\nu}{2l} \quad ...(10)$$

where $n = 1, 2, 3, \ldots$.

If Y is the value of Young's modulus and ρ is the density of the material of the bar, the velocity of the longitudinal waves is given by $v = \sqrt{Y/\rho}$ and hence the frequency of vibration is given by

$$\nu = \frac{n}{2l}\sqrt{\frac{Y}{\rho}} \quad ...(11)$$

Thus longitudinal stationary waves are formed in the bar for certain discrete values of frequency [$\nu = v/2l, v/l, 3v/2l, \ldots$, *when* $n = 1, 2, 3, \ldots$ etc.]. At these frequencies, the rod is said to be in resonance and these special frequencies are called ***resonant frequencies***.

Harmonics and Overtones

In case of a free-free bar, the standing waves are produced, when the frequency of longitudinal vibrations in the bar is given by

$$\therefore \quad \nu = \frac{n\nu}{2l} = \frac{n}{2l}\sqrt{\frac{Y}{\rho}} \qquad ...(12)$$

where $n = 1, 2, 3...$.

When $n = 1$, the lowest possible frequency, given by

$$\nu_1 = \frac{\nu}{2l} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}} \qquad \textit{(first harmonic)} \qquad ...(13)$$

is called the **fundamental frequency** of vibration of the bar.

The other frequencies with which standing waves can be formed in the bar are

$$\nu_2 = \frac{2\nu}{2l} = \frac{2}{2l}\sqrt{\frac{Y}{\rho}} = 2\nu_1 \qquad \textit{(2nd harmonic)} \qquad ...(14)$$

$$\nu_3 = \frac{3\nu}{2l} = \frac{3}{2l}\sqrt{\frac{Y}{\rho}} = 3\nu_1 \qquad \textit{(3rd harmonic)} \qquad ...(15)$$

etc. Thus possible allowed frequencies of vibration of the bar are an integral multiple of the fundamental frequency. But integral multiples of a frequency are called harmonics, hence the frequencies ν_1, ν_2, ν_3, ..., which are in the ratio 1 : 2 : 3 : ..... with the fundamental frequency, represent the **harmonic series**. Fundamental frequency (ν_1) is the **first harmonic** and ν_2, ν_3, ... are the **2nd harmonic**, **3rd harmonic**, ... respectively.

The frequencies with which the bar can vibrate except the fundamental frequency (fundamental tone) are called **overtones.** Thus $\nu_2 = 2\nu_1$ is the first overtone, $\nu_3 = 3\nu_1$, the second overtone etc.

Normal Modes of Vibration

In a normal mode of vibration of an oscillating system, all its particles vibrate sinusoidally with the same frequency. When the bar vibrates according to eq. (5) with some natural frequency, we say that it is vibrating in a normal mode. For *n*th normal mode of vibration,

$$k = \frac{n\pi}{l} \qquad [\text{(using eq. (6)}] \quad ...(16a)$$

and

$$\omega = 2\pi\nu = \frac{n\pi\nu}{l} \qquad [\text{using eq. (10)}] \quad ...(16b)$$

The frequencies, given by $\nu = n\nu/2l$, for $n = 1, 2, 3, ...$are called **normal mode frequencies** or simply **normal frequencies** in different modes of vibration of the free-free bar.

Hence in the *n*th normal mode of vibration of a bar, the equation for displacement at any time *t* at distance *x* from the origin is given by [using eq. (5)]:

$$y_n = A_n \cos\frac{n\pi x}{l}\sin\frac{n\pi\nu t}{l} \qquad ...(17)$$

When $n = 1$, the bar is said to vibrate in *fundamental mode* or *first normal mode* of vibration with frequency given by eq. (13). In such a case, the equation of standing wave will be

$$y_1 = A_1 \cos\frac{\pi x}{l}\sin\frac{\pi\nu t}{l} \qquad ...(18)$$

At any point *x*, the amplitude of oscillation of a particle is $A_1 \cos(\pi x/l)$. At $x = 0$ or $x = l$, the value of the amplitude ($\pm A_1$) is maximum. It is zero at $x = l/2$. Thus in the fundamental mode of vibration of the bar, free at both ends, we have antinodes (*A*) at the ends and one node (*N*) in the middle [**fig. 3 (a)**].

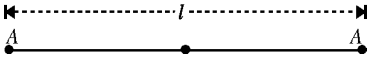
When $n = 2$, the bar vibrates in *second normal mode* of vibration with frequency given by eq. (14). The equation of standing wave in this mode is

$$y_2 = A_2 \cos\frac{2\pi x}{l}\sin\frac{2\pi\nu t}{l} \qquad ...(19)$$

Now at $x = 0$, $l/2$ and l, the value of the amplitude will be maximum and it is zero at $x = l/4$ and $3l/4$.

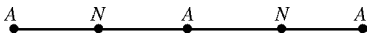
Thus in the second mode of vibration, in addition to the two antinodes (A) at the ends of the bar, one antinode (A) will be formed in the middle ($x = l/2$) and there will be two nodes (N) in between them [**fig. 3 (b)**].

Fundamental Tone
First Harmonic



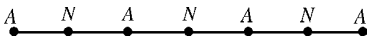
(a) Mode 1 or Fundamental Mode: $\lambda_1 = 2l,\ \nu_1 = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$

2nd Harmonic
First Overtone



(b) Mode 2: $\lambda_2 = l,\ \nu_2 = \frac{v}{l} = \frac{2}{2l}\sqrt{\frac{Y}{\rho}} = 2\nu_1$

3rd Harmonic
2nd Overtone



(c) Mode 3: $\lambda_3 = \frac{2l}{3},\ \nu_3 = \frac{3v}{2l} = \frac{3}{2l}\sqrt{\frac{Y}{\rho}} = 3\nu_1$

Fig. 3: Normal modes of free-free bar.

In the third mode of vibration ($n = 3$), the equation of standing wave is

$$y_3 = A_3 \cos \frac{3\pi x}{l} \sin \frac{3\pi \nu t}{l} \qquad ...(20)$$

In this mode, the antinodes (A) will be at $x = 0,\ l/3,\ 2l/3$ and l and the nodes (N) at $x = l/6,\ l/2,\ 5l/6$ [**fig. 3 (c)**].

Similarly in the nth mode of vibration free-free of the bar there will be $(n + 1)$ antinodes and n nodes.

General Vibration of the Free-free Bar and the Complete Solution

The general solution of the wave equation, given by eq. (17) i.e.,

$$y_n = A_n \cos \frac{n\pi x}{l} \sin \frac{n\pi \nu t}{l} \qquad ...(21)$$

represents the nth mode of vibration of a free-free bar with frequency $n\nu/2l$. As the wave equation is linear, the complete solution or the most general solution is obtained by the linear superposition of all the y_n i.e.,

$$y = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} \sin\frac{n\pi \nu t}{l} \qquad ...(22)$$

This represents the general longitudinal vibration of the bar, resulting from the superposition of all the modes of different frequencies.

6. Longitudinal Vibrations in a Bar, Fixed at Both Ends (Fixed-fixed Bar)

We write the general solution of the wave equation of the longitudinal vibrations in the form:

$$y = A \cos\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right) \qquad ...(1)$$

In the case of a bar fixed at both ends, there will be no displacement of particles at the ends of the bar ($x = 0$ and $x = l$). Thus, the boundary conditions are:

(i) $y = 0$ at $x = 0$ for all t, (ii) $y = 0$ at $x = l$ for all t

Eq. (1) is

$$y = A \cos\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right) \qquad ...(2)$$

In case of boundary condition (i), we obtain

$$0 = \cos \phi/2 \text{ or } \phi/2 = \pi/2 \text{ or } \phi = \pi \qquad ...(3)$$

Hence eq. (2) takes the form

$$y = A \cos\left(kx + \frac{\pi}{2}\right) \sin\left(\omega t + \frac{\pi}{2}\right) = -A \sin kx \cos \omega t$$

or $$y = B \sin kx \cos \omega t \quad ...(4)$$

where we have put $B = -A$.

For boundary condition (ii),

$$\sin kl = 0 \text{ or } kl = n\pi, \text{ where } n = 1, 2, 3, ... \quad ...(5)$$

Hence $$l = \frac{n\pi}{k} = \frac{n\pi}{2\pi/\lambda} = \frac{n\lambda}{2} \text{ or } \lambda = \frac{2l}{n} \quad ...(6)$$

Now, $$\text{frequency } \nu = \frac{v}{\lambda} = \frac{nv}{2l} = \frac{n}{2l}\sqrt{\frac{Y}{\rho}} \quad ...(7)$$

where Y = Young's modulus and ρ = density of the material of the bar.

Thus longitudinal standing waves are formed in the fixed-fixed bar for frequency $\nu_1 = v/2l$, $\nu_2 = v/l$ or $\nu_3 = 3v/2l$ etc. Thus ***free-free and fixed-fixed bars have identical frequencies.***

Harmonics and Overtones

The frequencies of harmonic series are obtained from relation (7), given by

$$\nu = \frac{nv}{2l} = \frac{n}{2l}\sqrt{\frac{Y}{\rho}} \quad ...(8)$$

where eq. (8) for $n = 1, 2, 3, ...$ represents first, second, third, ... harmonics respectively i.e.,

$$\nu_1 = \frac{1}{2l}\sqrt{\frac{Y}{\rho}} \qquad (\textit{fundamental or 1st harmonic})$$

$$\nu_2 = \frac{2v}{2l} = \frac{2}{2l}\sqrt{\frac{Y}{\rho}} = 2\nu_1 \quad (\textit{2nd harmonic or 1st overtone})$$

$$\nu_3 = \frac{3v}{2l} = \frac{3}{2l}\sqrt{\frac{Y}{\rho}} = 3\nu_1 \quad (\textit{3rd harmonic or 2nd overtone})$$

where ν_1 is the frequency of fundamental tone or first harmonic, ν_2 that of first overtone or second harmonic etc.

Normal Modes

Using eqs (4), (5) and (7) the nth normal mode of vibration of the bar is represented as

$$y_n = B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi vt}{l} \quad ...(9)$$

where $\omega = 2\pi\nu = n\pi v/l$ [using eq. (7)]. The frequencies, given by $\nu = nv/2l$ for $n = 1, 2, 3, ...$ are called the normal frequencies in different modes of vibration of the bar, fixed at both ends.

The equations of standing waves in different modes can be represented as

$$y_1 = B_1 \sin\frac{\pi x}{l} \cos\frac{\pi vt}{l} \qquad (\textit{fundamental mode or 1st normal mode})$$

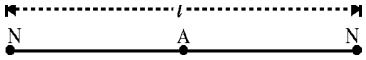
$$y_2 = B_2 \sin\frac{2\pi x}{l} \cos\frac{2\pi vt}{l} \quad (\textit{2nd normal mode})$$

$$y_3 = B_3 \sin\frac{3\pi x}{l} \cos\frac{3\pi vt}{l} \quad (\textit{3rd normal mode})$$

When the rod is vibrating in the fundamental mode, we find that at $x = 0$ and $x = l$, the amplitude is zero and at $x = l/2$ it is maximum ($\pm B_1$). Therefore, refix in the fundamental mode of vibration of the

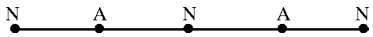
fixed-fixed bar, we obtain two nodes (N) at the ends and one antinode (A) at the middle [**fig. 4 (a)**] . In the second normal mode, at $x = 0, l/2, l$, three nodes (N) and at $x = l/4, 3l/4$, two antinodes are formed [**fig. 4 (b)**].

Fundamental
First Harmonic



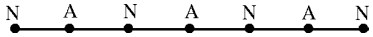
(a) Mode 1 or Fundamental Mode: $\lambda_1 = 2l,\ \nu_1 = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$

2nd Harmonic
1st Overtone



(b) Mode 2: $\lambda_2 = l,\ \nu_2 = \frac{v}{l} = \frac{1}{l}\sqrt{\frac{Y}{\rho}} = 2\nu_1$

3rd Harmonic
2nd Overtone



(c) Mode 3: $\lambda_3 = 2l/3,\ \nu_3 = \frac{3v}{2l} = \frac{3}{2l}\sqrt{\frac{Y}{\rho}} = 3\nu_1$

Fig. 4: Normal modes of fixed-fixed bar.

In the 3rd mode, there are four nodes (N) and three antinodes (A) in between them at equal distances [**fig. 4 (c)**]. Thus, if the bar is vibrating in the nth normal mode, there will be $(n + 1)$ nodes and n antinodes.

In case of a fixed-fixed bar, the general solution of the wave equation, given by eq. (9), represents the nth mode of vibration. The *complete solution* representing the general vibration, is obtained by the linear superposition of all the modes of different frequencies i.e.,

$$y = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi v t}{l} \qquad ...(10)$$

7. Longitudinal Vibrations in a Bar Fixed at the Mid-point

The general solution of the wave equation, representing stationary waves, for the longitudinal vibrations in a bar, is given by

$$y = A\cos\left(kx + \frac{\phi}{2}\right)\sin\left(\omega t + \frac{\phi}{2}\right) \qquad ...(1)$$

If the bar is fixed (or clamped) at the mid point, the staionary waves will be set up in such a manner that the mid point is always a node and the ends are antinodes. The boundary conditions :re:

(i) $\frac{dy}{dx} = 0$ at $x = 0$ and $x = l$ at all t

(ii) $y = 0$ at $x = l/2$ at all t.

Differentiating eq. (1) with respect to x, we get

$$\frac{dy}{dx} = -Ak\sin\left(kx + \frac{\phi}{2}\right)\sin\left(\omega t + \frac{\phi}{2}\right)$$

For boundary condition (i), similar to the case of free-free bar [Art. 10.5], we may obtain the equation of standing waves in the bar as [eq. (5), Art. 5]

$$y = A\cos kx\sin\omega t \qquad ...(2)$$

where $k = n\pi/l$ or $\lambda = 2\pi/k = 2l/n$ and $\omega = 2\pi\nu = 2\pi\frac{v}{\lambda} = \frac{n\pi v}{l} = \frac{n\pi}{l}\sqrt{\frac{Y}{\rho}}$.

If we apply boundary condition (ii), we obtain from eq. (2)

$$0 = \cos\frac{n\pi}{l}\cdot\frac{l}{2} \text{ or } \cos\frac{n\pi}{2} = 0$$

whence $\frac{n\pi}{2} = (2r - 1)\frac{\pi}{2}$, where $r = 1, 2, 3,...$

or $$n = 2r - 1 \text{ i.e., } n = 1, 3, 5,... \quad ...(3)$$

Thus, eq. (2) takes the form

$$y_n = A_n \cos \frac{n\pi x}{l} \sin \frac{n\pi vt}{l} \quad ...(4)$$

where $n = 1, 3, 5...$.

The frequencies of the harmonics ($\nu = \omega/2\pi$) are given by

$$\nu = \frac{n\nu}{2l} = \frac{n}{2l}\sqrt{\frac{Y}{\rho}} \quad ...(5)$$

Frequency of the fundamental note (or first harmonic) is obtained for $n = 1$ (or $r = 1$) i.e.,

$$\nu_1 = \frac{\nu}{2l} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}} \quad ...(6a)$$

If we put $n = 3$ (or $r = 2$) and $n = 5$ (or $r = 3$), the frequencies of the first and second overtones (or higher harmonics) are found to be

$$\nu_2 = \frac{3\nu}{2l} \frac{3}{2l}\sqrt{\frac{Y}{\rho}} = 3\nu_1 \text{ and } \nu_3 = \frac{5\nu}{2l} = \frac{5}{2l}\sqrt{\frac{Y}{\rho}} = 5\nu_1 \quad ...(6b)$$

Hence the frequencies of the harmonics are in the ratio 1 : 3 : 5 : ... . Thus we find that **when stationary waves are formed in a bar clamped at the mid point, only odd harmonics are found and even harmonics are absent**.

Writing $n = 2r - 1$ in eq. (4), we obtain the equation of rth mode of vibration of the bar i.e.,

$$y_r = A_r \cos (2r - 1) \frac{\pi x}{l} \sin (2r - 1) \frac{\pi vt}{l} \quad ...(7)$$

For $r = 1, 2, 3$, we obtain the first three normal modes of vibration of the bar fixed at the mid point and the corresponding equations of standing waves are obtained from eq. (4):

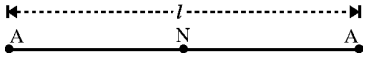
$$y_1 = A_1 \cos \frac{\pi x}{l} \sin \frac{\pi vt}{l} \quad \textit{(Fundamental mode)}$$

$$y_2 = A_2 \cos \frac{3\pi x}{l} \sin \frac{3\pi vt}{l} \quad \textit{(2nd mode)} \quad ...(8)$$

$$y_3 = A_3 \cos \frac{5\pi x}{l} \sin \frac{5\pi vt}{l} \quad \textit{(3rd mode)}$$

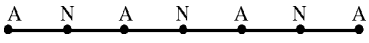
The first three modes of vibration of a bar fixed at the mid point and free at the two ends are shown in **fig. 5**. A mode is always formed at the fixed mid point and antinodes at the two free ends. In the fundamental mode of vibration, a node (N) is formed in the middle at $x = l/2$ and two antinodes (A) at the free ends ($x = 0$ and $x = l$). In the second mode of vibration, three nodes (N) and four antinodes (A) and in the third mode of vibration, 5 nodes (N) and six antinodes (A) are produced. Obviously in the rth mode of vibration of a bar, fixed at the mid point, there will be $(2r - 1)$ nodes and $2r$ antinodes.

Fundamental
First harmonic



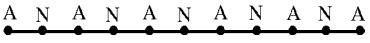
(a) Mode 1 or fundamental mode: $\lambda_1 = 2l$, $\nu_1 = \nu/2l$

First overtone
Third harmonic



(b) Mode 2: $\lambda_2 = 2l/3$, $\nu_2 = 3\nu/2l = 3\nu_1$

Second overtone
Fifth harmonic



(c) Mode 3: $\lambda_3 = 2l/5$, $\nu_3 = 5\nu/2l = 5\nu_1$

Fig. 5: Normal mdoes of a bar fixed at the mid point.

The general solution of the wave equation for a bar fixed at the mid point, given by eq. (7), represents the rth mode of vibration of the bar. Hence the complete solution, representing the general vibration, will be obtained as

$$y = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} A_n \cos (2n-1)\frac{\pi x}{l} \sin (2n-1)\frac{\pi v t}{l} \quad ...(9)$$

8. Longitudinal Vibrations in a Bar Fixed at One End and Free at the Other (Fixed-free Bar)

The general solution of the wave equation for the longitudinal vibrations may represent stationary waves and is taken in the form

$$y = A \cos\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right) \quad ...(1)$$

When the standing waves are formed in a bar, fixed at one end and free at the other, a node is formed at the fixed end ($x = 0$) and an antinode on the free end ($x = l$). In such a case, the boundary conditions are:

(i) $y = 0$ at $x = 0$ for all t

(ii) $\frac{dy}{dx} = 0$ at $x = l$ for all t ...(2)

Using boundary condition (i), we get

$$\cos \phi/2 = 0 \text{ or } \phi/2 = \pi/2 \text{ or } \phi = \pi \quad ...(3)$$

Hence $$y = -2a \sin kx \cos \omega t \quad ...(4)$$

Now $$\frac{dy}{dx} = -2ak \cos kx \cos \omega t$$

Using boundary condition (ii),

$$\cos kl = 0 \text{ or } kl = (2n-1)\frac{\pi}{2} \quad ...(5)$$

where $n = 1, 2, 3,...$.

$$\therefore \quad k = (2n-1)\frac{\pi}{2l} \text{ or } \lambda = \frac{2\pi}{k} = \frac{4l}{2n-1} \quad ...(6)$$

or $$l = \frac{(2n-1)\lambda}{4} \quad ...(7)$$

Thus, in case of a bar fixed at one end only, standing waves are formed when the length of the rod is an odd multiple of $\lambda/4$.

Now, $$\text{frequency, } \nu = \frac{v}{\lambda} = \frac{(2n-1)v}{4l} = \frac{(2n-1)}{4l}\sqrt{\frac{Y}{\rho}} \quad ...(8)$$

where $n = 1, 2, 3,...$.

Thus, longitudinal stationary waves are set up in the bar for certain discrete frequencies, corresponding to $n = 1, 2, 3,...$

For $n = 1$, $$\nu_1 = \frac{v}{4l} = \frac{1}{4l}\sqrt{\frac{Y}{\rho}} \text{ (Fundamental tone/first harmonic)} \quad ...(9a)$$

For $n = 2$, $$\nu_2 = \frac{3v}{4l} = \frac{3}{4l}\sqrt{\frac{Y}{\rho}} = 3\nu_1 \text{ (First overtone/third harmonic)} \quad ...(9b)$$

For $n = 3$, $$\nu_3 = \frac{5v}{4l} = \frac{5}{4l}\sqrt{\frac{Y}{\rho}} = 5\nu_1 \text{ (Second overtone/fifth harmonic)} \quad ...(9c)$$

We see that the frequencies of fundamental, first overtone, second overtone etc. are in the ratio 1 : 3 : 5 : ... etc. and hence ***for a bar fixed at one end, only odd harmonics are found and even harmonics are absent.***

First three normal modes of vibration are obtained for $n = 1, 2, 3$ and the corresponding equations of standing waves are given by eq. (4) i.e.,

$$y = -2a \sin kx \cos \omega t$$

where $k = (2n-1)\frac{\pi}{2l}$ and $\omega = 2\pi\nu = \frac{(2n-1)\pi v}{2l}$

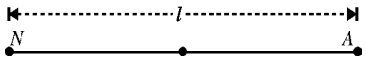
Thus, $$y = -2a \sin (2n-1)\frac{\pi x}{2l} \cos \frac{(2n-1)\pi v t}{2l} \quad ...(10)$$

For $n = 1$, $y_1 = -2a \sin \frac{\pi x}{2l} \cos \frac{\pi v t}{2l}$ *(Fundamental or first mode)* ...(11a)

For $n = 2$, $y_2 = -2a \sin \frac{3\pi x}{2l} \cos \frac{3\pi v t}{2l}$ *(Second mode)* ...(11b)

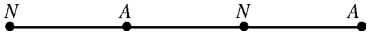
For $n = 3$, $y_3 = -2a \sin \frac{5\pi x}{2l} \cos \frac{5\pi v t}{2l}$ *(Third mode)* ...(11c)

Fundamental tone
First harmonic



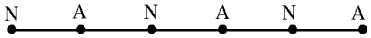
(a) Mode 1 or fundamental mode: $\lambda = 4l$, $\nu_1 = \frac{v}{4l} = \frac{1}{4l}\sqrt{\frac{Y}{\rho}}$

First overtone
3rd harmonic



(b) Mode 2: $\lambda_2 = \frac{4l}{3}$, $\nu_2 = \frac{3v}{4l} = \frac{3}{4l}\sqrt{\frac{Y}{\rho}} = 3\nu_1$

Second overtone
5th harmonic



(c) Mode 3: $\lambda_3 = \frac{4l}{5}$, $\nu_3 = \frac{5v}{4l} = \frac{5}{4l}\sqrt{\frac{Y}{\rho}} = 5\nu_1$

Fig. 6: Normal modes of a bar fixed at one end.

Fig. 6 shows first three normal modes of vibration of a bar fixed at one end and free at the other end. In the fundamental mode of vibration, a node is formed at the fixed end and an antinode at the free end at $x = l$. In the second mode of vibration, two nodes are formed at $x = 0$ and $x = 2l/3$ and two antinodes at $x = l/3$ and $x = l$. In the third normal mode, three nodes are obtained at $x = 0$, $2l/5$ and $4l/5$ and three antinodes at $x = l/5$, $3l/5$ and l. Thus in case of a bar fixed at one end, nth mode of vibration will have n nodes and n antinodes.

The general solution of the wave equation for a fixed free bar, given by eq. (10), represents the nth mode of vibration. The general vibration of the bar is represented by the complete solution of the wave equation and is obtained by the linear superposition of all the modes of different frequencies i.e.,

$$y = \sum_{n=1}^{\infty} (-2a)\sin(2n-1)\frac{\pi x}{2}\cos(2n-1)\frac{\pi v t}{2l} \quad ...(12)$$

9. Transverse Vibrations of a Bar – Differential Wave Equation

Transverse vibrations can be produced in a bar or rod of elastic material. If a bar is clamped at one end and displaced at the free end through a small distance, then on releasing, the bar vibrates transversely. Transverse vibrations can also be produced in a free-free (free at both ends) bar by holding at the two ends and striking a blow near the middle.

When a bar executes transverse vibrations, the strains are elongation and contraction of the filaments of the rod on the two sides of the neutral axis. Hence the forces which come into play are those appearing

---

* In case of a rod of rectangular cross-section of breadth b and thickness (depth) d, $I_g = \frac{bd^3}{12}$ and $A = bd$;

$\therefore$ $I_g = AK^2 = bdK^2 = bd^3/12$, whence $K = d/2\sqrt{3}$.

For a rod of circular cross-section of radius r; $I_g = \pi r^4/4$ and $I_g = AK^2 = \pi r^2 K^2$; $\therefore$ $\pi r^2 K^2 = \pi r^4/4$ or $K = r/2$.

in the bending of a beam. If we apply a force F perpendicular to the neutral axis, the bending takes place and the bending moment M is given by

$$M = \frac{YI_g}{R} \qquad ...(1)$$

where Y is the Young's modulus of the material of the bar and $I_g = AK^2$ is the geometrical moment of intertia of the section of the bar (A = area of cross-section of the bar and K = geometrical radius of gyration of the area of cross-section about the neutral plane)*. Here, R is the radius of curvature of the netural surface, given by

$$R = \frac{\{1+(\partial y/\partial x)^2\}^{3/2}}{\partial^2 y/\partial x^2}.$$

The X-axis is taken along the neutral axis and y is the displacement at the point x [**fig. 7**].

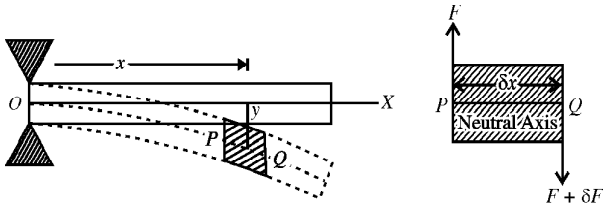


Fig. 7: Transverse vibrations of a bar.

For small y, dy/dx is also very small and hence neglecting $(dy/dx)^2$ in comparison to 1, we obtain $R = 1/(\partial^2 y/\partial x^2)$. Therefore eq. (1) takes the form

$$M = YAK^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(2)$$

Further the force F is related to the bending moment as

$$F = \frac{\partial M}{\partial x} \qquad ...(3)$$

Let us now consider an element PQ of length δx. The faces of the element, P and Q, are perpendicular to the neutral axis. Let F be the shearing force at the face P. Then the shearing force at the face Q is $F + \delta F = F + \frac{\partial F}{\partial x}\delta x$. Then the unbalanced force on the element is $-\frac{\partial F}{\partial x}\delta x$. If y be the displacement of the element at time t, then according to Newton's second law of motion, this force can be expressed as the product of mass ($m = \rho A \delta x$) and acceleration ($a = d^2y/dt^2$) i.e.,

$$(\rho A \delta x)\frac{\partial^2 y}{\partial t^2} = -\frac{\partial F}{\partial x}\delta x \qquad ...(4)$$

where ρ is the density of the material of the element (bar) and A is the area of cross-section of the bar.

Eq. (4) gives

$$\rho A \frac{\partial^2 y}{dt^2} = -\frac{\partial F}{\partial x} \qquad ...(5)$$

On using eq. (3) and eq. (2) we obtain

$$\rho A \frac{\partial^2 y}{\partial t^2} = -\frac{\partial}{\partial x}\left(\frac{\partial M}{\partial x}\right) = -\frac{\partial^2}{\partial x^2}\left(YAK^2\frac{\partial^2 y}{\partial x^2}\right)$$

or $$\rho A \frac{\partial^2 y}{\partial t^2} = -YAK^2\frac{\partial^4 y}{\partial x^4} \text{ or } \frac{\partial^4 y}{\partial x^4} = \frac{\rho}{YK^2}\cdot\frac{\partial^2 y}{\partial t^2}$$

or $$\frac{\partial^4 y}{\partial x^4} + \frac{\rho}{YK^2}\cdot\frac{\partial^2 y}{\partial t^2} = 0 \qquad ...(6)$$

This is the ***differential wave equation*** **for the transverse waves** in a bar or rod.

10. General Solution of the Wave Equation for Transverse Vibrations

The wave equation for transverse vibrations of a bar is given by

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho}{AK^2}\frac{\partial^2 y}{\partial t^2} = 0 \qquad ...(1)$$

Let the bar be executing simple harmonic transverse vibrations. Suppose that the bar vibrates with an amplitude $a(x)$, where a is a function of x, so that the solution is of the form

$$y = a(x) \sin \omega t \qquad ...(2)$$

where ω is the angular frequency of vibration.

Hence, $$\frac{\partial^2 y}{\partial t^2} = -\omega^2 a(x) \sin \omega t \qquad ...(3a)$$

and $$\frac{\partial^4 y}{\partial x^4} = \frac{d^4 a(x)}{dx^4} \sin \omega t \qquad ...(3b)$$

Hence, eq. (1) takes the form

$$\frac{d^4 a(x)}{dx^4} - \frac{\rho\omega^2}{YK^2} a(x) = 0$$

or $$\frac{d^4 a(x)}{dx^4} - \alpha^4 a(x) = 0 \qquad ...(4)$$

where $$\alpha^4 = \rho\omega^2/YK^2 \qquad ...(5)$$

To solve eq. (4), we put $\frac{d^4}{dx^4} = D^4$. Then, eq. (4) takes the form

$$D^4 a(x) - \alpha^4 a(x) = 0 \quad \text{or} \quad (D^4 - \alpha^4) a(x) = 0$$

or $$(D^2 - \alpha^2)(D^2 + \alpha^2) a(x) = 0$$

Therefore, either $(D^2 - \alpha^2) a(x) = 0$ or $(D^2 + \alpha^2) a(x) = 0$

i.e., either $$\frac{d^2 a(x)}{dx^2} - \alpha^2 a(x) = 0 \quad \text{or} \quad \frac{d^2 a(x)}{dx^2} + \alpha^2 a(x) = 0 \qquad ...(6)$$

The solution of $\frac{d^2 a(x)}{dx^2} - \alpha^2 a(x) = 0$ is

$$a(x) = A \cosh \alpha x + B \sinh \alpha x \qquad ...(7)$$

Also the solution of $\frac{d^2 a(x)}{dx^2} + \alpha^2 a(x) = 0$ is

$$a(x) = C \cos \alpha x + D \sin \alpha x \qquad ...(8)$$

The general solution is the linear combination of the solutions (7) and (8). Hence

$$a(x) = A \cosh \alpha x + B \sinh \alpha x + C \cos \alpha x + D \sin \alpha x \qquad ...(9)$$

The complete general solution of the wave equation (1) is obtained by putting for $a(x)$ from (9) into eq. (2) i.e.,

$$y = (A \cosh \alpha x + B \sinh \alpha x + C \cos \alpha x + D \sin \alpha x) \sin \omega t \qquad ...(10)$$

where A, B, C and D are arbitary constants to be determined from the boundary conditions.

Boundary Conditions

(1) At the Fixed (or Clamped) End:

$y = 0$ for all t, because there is no displacement at the fixed end,

$\frac{dy}{dx} = 0$ for all t, because there is no slope at the fixed end.

(2) At the Free End:

$\frac{d^2y}{dx^2} = 0$ for all t, because there is no bending moment at the free end i.e., $M = YAK^2 d^2y/dx^2 = 0$.

$\frac{d^3y}{dx^3} = 0$ for all t, because there is no force at the free end i.e., $F = \frac{dM}{dx} = YAK^2 \frac{d^3y}{dx^3} = 0$.

11. Transverse Vibrations in a Bar Fixed at One End and Free at the Other (Clamped Free Bar)

The general solution of the wave equation for transverse waves in a bar is given by

$$y = (A \cosh \alpha x + B \sinh \alpha x + C \cos \alpha x + D \sin \alpha x) \sin \omega t \quad ...(1)$$

where A, B, C, D are constants and $\alpha = (\rho\omega^2/YK^2)^{1/4}$ (ρ = density of the material of the rod, Y = Young's modulus, K = geometrical radius of gyration of the area of cross-section of the bar about neutral plane and ω = angular frequency).

Let the bar be clamped at the end $x = 0$ and free at the end $x = l$. The *boundary conditions* are:

(i) At $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$ for all t.

(ii) At $x = l$, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} = 0$ for all t.

Differentiating eq. (1) with respect to x and further differentiations yield

$$\frac{dy}{dx} = (A\alpha \sinh \alpha x + B\alpha \cosh \alpha x - C\alpha \sin \alpha x + D\alpha \cos \alpha x) \sin \omega t \quad ...(2)$$

$$\frac{d^2y}{dx^2} = (A\alpha^2 \cosh \alpha x + B\alpha^2 \sinh \alpha x - C\alpha^2 \cos \alpha x - D\alpha^2 \sin \alpha x) \sin \omega t \quad ...(3)$$

$$\frac{d^3y}{dx^3} = (A\alpha^3 \sinh \alpha x + B\alpha^3 \cosh \alpha x + C\alpha^3 \sin \alpha x - D\alpha^3 \cos \alpha x) \sin \omega t \quad ...(4)$$

If we put the condition (i) in eqs. (1) and (2), we obtain

$$A + C = 0 \text{ and } B + D = 0$$

Hence $$C = -A \text{ and } D = -B \quad ...(5)$$

Putting condition (ii) in eqs. (3) and (4), we get

$$A(\sinh \alpha l + \cos \alpha l) + B(\sinh \alpha l + \sin \alpha l) = 0 \quad ...(6)$$

$$A(\cosh \alpha l - \sin \alpha l) + B(\cosh \alpha l + \cos \alpha l) = 0 \quad ...(7)$$

For nonvanishing A and B, the determinant of eqs. (6) and (7) vanishes i.e.,

$$\begin{vmatrix} \cosh \alpha l + \cos \alpha l & \sinh \alpha l + \sin \alpha l \\ \sinh \alpha l - \sin \alpha l & \cosh \alpha l + \cos \alpha l \end{vmatrix} = 0$$

or $(\cosh \alpha l + \cos \alpha l)^2 - (\sinh^2 \alpha l - \sin^2 \alpha l) = 0$

or $2 + 2 \cosh \alpha l \cos \alpha l = 0$ [$\because \cosh^2 \alpha l - \sinh^2 \alpha l = 1$ and $\cos^2 \alpha l + \sin^2 \alpha l = 1$]

or $(\cosh \alpha l + \cos \alpha l = -1$

or $$\cosh \alpha l = -\sec \alpha l \quad ...(8)$$

The solution of the eq. (8) is obtained by plotting the graph of the two sides of the eq. (8) i.e., $\cosh \alpha l$ and $-\sec \alpha l$ against the angle αl and locating the points of intersection of the two curves [**fig. 8**]. The solutions are obtained to be

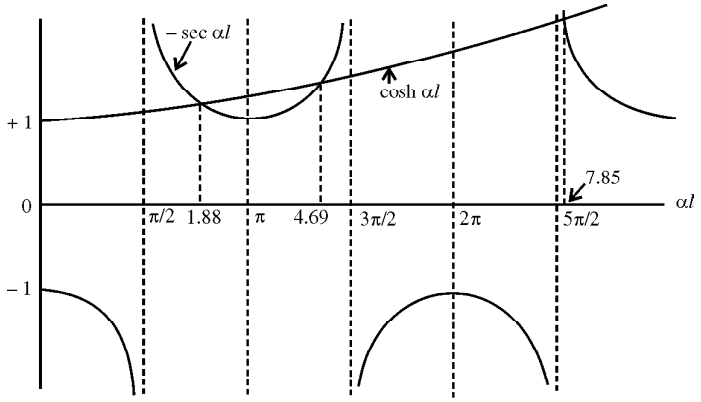


Fig. 8: Graphical solution of eq. (8)

$$\alpha l = 1.88, 4.69, 7.85, \ldots\ldots \qquad ...(9)$$

which are close to $\pi/2$, $3\pi/2$, $5\pi/2$, ... , but are exactly equal at higher values.

In eq. (1), the expression for α is

$$\alpha = \left(\frac{\rho\omega^2}{YK^2}\right)^{1/4} \quad \text{or} \quad \alpha^4 = \frac{\rho\omega^2}{YK^2}$$

Hence $$\omega^2 = \frac{\alpha^4 K^2 Y}{\rho} \quad \text{or} \quad \omega = \alpha^2 K\sqrt{\frac{Y}{\rho}} = \alpha^2 K v \qquad ...(10)$$

where $v = \sqrt{Y/\rho}$ is the velocity of longitudinal waves.

Eq. (10) represents the expression for the *angular frequency* of vibration. Hence the **frequency of vibration** is given by

$$v = \frac{\omega}{2\pi} = \frac{\alpha^2 K v}{2\pi} = \frac{\alpha^2 K}{2\pi}\sqrt{\frac{Y}{\rho}} \qquad ...(11)$$

where $\alpha = \frac{1.88}{l}, \frac{4.69}{l}, \frac{7.85}{l}, \ldots\ldots$.

For a bar of thickness d, $K = d/\sqrt{12}$ and for a bar of radius r, $K = r/2$.

Thus the bar can vibrate in different modes for different values of α, but here upper modes or overtones will not have frequencies in exact integral multiples of the lowest mode (for $\alpha = 1{\cdot}88/l$). The lowest mode is called *fundamental mode of vibration.* Thus **the frequencies of overtones are not harmonics of fundamental frequency**. First three modes of vibration of a clamped free bar are shown in **fig. 9**. Remember that when upper modes have frequencies in exact integral multiples of the frequency of the fundamental mode, they are called *harmonics*, otherwise they are called *overtones* or *upper partials.*

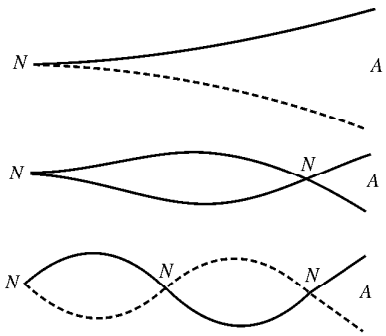


Fig. 9: First three modes of vibration of clamped free bar.

From eq. (11), the frequency of the ***fundamental*** is given by

$$v_1 = \frac{(1.88)^2}{l^2} \cdot \frac{Kv}{2\pi} = \frac{(1.88)^2}{l^2} \cdot \frac{K}{2\pi}\sqrt{\frac{Y}{\rho}} \qquad ...(12)$$

The frequencies of the **first and second overtones** are given by

$$v_2 = \frac{(4.69)^2}{l^2} \cdot \frac{Kv}{2\pi} = \frac{(4.69)^2}{l^2} \cdot \frac{K}{2\pi}\sqrt{\frac{Y}{\rho}} \qquad ...(13)$$

and $$\nu_3 = \frac{(7.85)^2}{l^2}\cdot\frac{K\nu}{2\pi} = \frac{(7.85)^2}{l^2}\cdot\frac{K}{2\pi}\sqrt{\frac{Y}{\rho}} \quad ...(14)$$

Thus $\nu_1 : \nu_2 : \nu_3 = 1 : 6.27 : 17.55$ and hence the partials do not form a harmonic series. Therefore, the sound emitted by the vibrating bar transversely is *nonmelodius.* This is why in the musical instruments, the vibrating bars or reeds are seldomly used.

12. Transverse Vibrations in a Bar Free at Both Ends (Free-free Bar)

We write the general solution of the wave equation for transverse waves in a bar in the form

$$y = (A \cosh \alpha x + B \sinh \alpha x + C \cos \alpha x + D \sin \alpha x) \sin \omega t \quad ...(1)$$

where A, B, C, D are constants and $\alpha = (\rho\omega^2/YK^2)^{1/4}$.

When the bar is free at both ends i.e., at $x = 0$ and $x = l$, the *boundary conditions* are:

(i) At $x = 0$, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} = 0$ for all t.

(ii) At $x = l$, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} = 0$ for all t.

From (1), differentiating y with respect to x two times and three times, we get

$$\frac{d^2y}{dx^2} = (A\alpha^2 \cosh \alpha x + B\alpha^2 \sinh \alpha x - C\alpha^2 \cos \alpha x - D\alpha^2 \sin \alpha x) \sin \omega t \quad ...(2)$$

$$\frac{d^3y}{dx^3} = (A\alpha^3 \sinh \alpha x + B\alpha^3 \cosh \alpha x + C\alpha^3 \sin \alpha x - D\alpha^3 \cos \alpha x) \sin \omega t \quad ...(3)$$

Using condition (i) in eqs. (2) and (3), we get

$$A - C = 0 \text{ and } B - D = 0$$

or $$C = A \text{ and } D = B \quad ...(4)$$

Applying condition (ii), in eqs. (2) and (3), we obtain

$$A(\cosh \alpha l - \cos \alpha l) = B(\sin \alpha l - \sinh \alpha l) \quad ...(5)$$

$$A(\sinh \alpha l + \sin \alpha l) = B(\cos \alpha l - \cosh \alpha l) \quad ...(6)$$

Dividing eq. (5) by eq. (6),

$$\frac{\cosh \alpha l - \cos \alpha l}{\sinh \alpha l + \sin \alpha l} = \frac{\sin \alpha l - \sinh \alpha l}{\cos \alpha l - \cosh \alpha l}$$

$$(\cosh \alpha l - \cos \alpha l)^2 = \sinh^2 \alpha l - \sin^2 \alpha l$$

Using $\cosh^2 \alpha l - \sinh^2 \alpha l = \cos^2 \alpha l + \sin^2 \alpha l = 1$, we get

$$2 - 2 \cosh \alpha l \cos \alpha l = 0$$

or $$\cosh \alpha l = \sec \alpha l \quad ...(7)$$

The graphical solution of eq. (7) gives

$$\alpha l = 4.73, 7.85, 10.996, ... \quad ...(8)$$

Now from the relation $\alpha = (\rho\omega^2/YK^2)^{1/4}$,

we get the expressions for angular frequency (ω) and frequency (ν) as:

$$\omega = \alpha^2 K\sqrt{\frac{Y}{\rho}} = \alpha^2 K\nu \text{ and } \nu = \frac{\alpha^2 K}{2\pi}\sqrt{\frac{Y}{\rho}} = \frac{\alpha^2 K\nu}{2\pi} \quad ...(9)$$

where $v = \sqrt{Y/\rho}$ is the velocity of longitudinal waves in the bar.

Hence the frequency of the fundamental mode is

$$\nu_1 = \frac{(4.73)^2}{l^2}\cdot\frac{K}{2\pi}\sqrt{\frac{Y}{\rho}} = \frac{(4.73)^2}{l^2}\cdot\frac{K\nu}{2\pi} \quad ...(10)$$

The frequencies of the first two overtones are:

$$\nu_2 = \frac{(7.85)^2}{l^2}\cdot\frac{K}{2\pi}\sqrt{\frac{Y}{\rho}} = \frac{(7.85)^2}{l^2}\cdot\frac{K\nu}{2\pi}$$

and $$\nu_3 = \frac{(10.996)^2}{l^2}\cdot\frac{K}{2\pi}\sqrt{\frac{Y}{\rho}} = \frac{(10.996)^2}{l^2}\cdot\frac{K\nu}{2\pi} \quad ...(11)$$

Hence the frequencies of the first three modes of vibration of a free-free bar are in the ratio $\nu_1 : \nu_2 : \nu_3 = 1 : 2.76 : 5.40$.

Thus, *the partials for the transverse vibrations of a free-face bar do not give a harmonic series.* These modes of the free-free bar are shown in **fig. 10**.

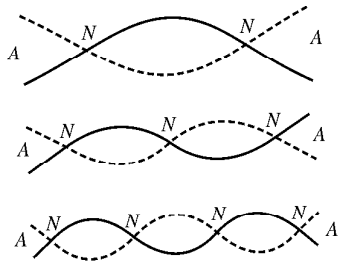


Fig. 10: First three modes of free-free bar.

Comparison of Frequencies for Longitudinal and Transverse Virbations

We consider the frequencies for longitudinal and transverse vibrations, when a clamped free bar vibrates in fundamental mode.

Frequency of longitudinal vibrations in fundamental mode :

$$\nu_l = \frac{v}{4l} = \frac{1}{4l}\sqrt{\frac{Y}{\rho}} \quad ...(12)$$

Frequency of transverse vibrations in fundamental mode:

$$\nu_t = \frac{(1.88)^2}{l^2}\cdot\frac{K}{2\pi}\sqrt{\frac{Y}{\rho}} \quad ...(13)$$

Hence, $$\frac{\nu_t}{\nu_l} = \frac{2.24K}{l} \quad ...(14)$$

Since the value of K ($K = r/2$ for cylindrical bar and $K = d/\sqrt{12}$ for a bar of thickness d), is much less than l, *the frequency of transverse vibrations is much less than the frequency of longitudinal vibrations for a bar when in both cases fundamental mode of vibration is considered.*

13. Velocity of Transverse Waves in a Bar

The differential wave equation for the transverse waves in a rod is given by

$$\frac{\partial^4 y}{\partial x^4} + \left(\frac{\rho}{YK^2}\right)\frac{\partial^2 y}{\partial t^2} = 0 \quad ...(1)$$

Let the solution of the wave equation be in the plane wave form:

$$y = ae^{i(\omega t - kx)} \quad ...(2)$$

where ω is the angular frequency and $k = 2\pi/\lambda$ is the wave number.

From (2) $$\frac{\partial^4 y}{\partial x^4} = k^4 y \text{ and } \frac{\partial^2 y}{\partial t^2} = -\omega^2 y$$

Putting the values of $\partial^4 y/\partial x^4$ and $\partial^2 y/\partial t^2$ in eq. (1), we obtain

$$k^4 y - \left(\frac{\rho}{YK^2}\right)\omega^2 y = 0 \quad ...(3)$$

The wave velocity (u) is given by

$$u = \frac{\omega}{k} \quad ...(4)$$

Hence $$k = \frac{\omega}{u} \quad ...(5)$$

Substituting $k = \omega/v$ in eq. (3), we get

$$\frac{\omega^4}{u^4} - \left(\frac{\rho}{YK^2}\right)\omega^2 = 0 \text{ or } u^4 = \frac{\omega^2 Y}{\rho}K^2$$

Since $v = \sqrt{Y/\rho}$ represents the velocity of longitudinal waves in the bar, we have

$$u^4 = \omega^2 v^2 K^2$$

whence $$u = \sqrt{\omega v K} \quad ...(6)$$

This is the *expression for the wave velocity* for the transverse vibration of a bar.

From eq. (6), we see that the wave velocity (u) is not constant but a function of angular frequency ω. Hence the medium is *dispersive* for transverse waves in rods.

From eq. (4), if we put $\omega = ku$, then

$$u = \sqrt{kuvK} \text{ or } \sqrt{u} = \sqrt{kvK}$$

or $$u = kvK = \frac{2\pi K}{\lambda}\sqrt{\frac{Y}{\rho}} \quad ...(7)$$

This is another form of the expression for the wave velocity of transverse waves in a bar.

Also from (4) and (7),

$$\frac{\omega}{k} = kvK \text{ or } \omega = k^2 Kv \quad ...(8)$$

If a number of frequencies are present in the vibrating bar, the group velocity (u_g) is given by

$$u_g = \frac{d\omega}{dk} = 2kKv = \frac{4\pi K}{\lambda}\sqrt{\frac{Y}{\rho}} \quad ...(9)$$

14. Comparision of Transverse Vibrations of Bar and String

(1) When a bar vibrates transversely, this is the elasticity of the material of the bar which is responsible to restore the successive portions of the bar to their original position. However, in case of string it is the tension in the string which is responsible for transverse vibrations.

(2) In case of a string, the velocity of transverse waves is given by $v = \sqrt{Y/\rho}$ and does not depend on the wavelength, but the velocity of the transverse waves in a bar is given by $v = \frac{2\pi K}{\lambda}\sqrt{\frac{Y}{\rho}}$ and hence depends on the wavelength λ.

15. Comparison of Longitudinal and Transverse Vibrations in a Bar

S. No.	Longitudinal Vibrations	Transverse Vibrations
1.	Differential wave equation $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$ where $v = \sqrt{Y/\rho}$.	Differential wave equation $\frac{\partial^2 y}{\partial t^2} = -v^2 K^2 \frac{\partial^4 y}{\partial x^4}$ where $v = \sqrt{Y/\rho}$ and K = geometrical radius of gyration.
2.	Velocity of longitudinal waves in a bar: $v = \sqrt{\frac{Y}{\rho}}$ where Y = young's modulus and ρ = density of material of bar.	Velocity of transverse waves in a bar: $u = \sqrt{\omega v K} = \frac{2\pi}{\lambda} Kv = \frac{2\pi K}{\lambda}\sqrt{\frac{Y}{\rho}}$ where ω = angular frequency, v = velocity of longitudinal waves and λ = wavelength.
3.	Velocity of longitudinal waves in a bar is constant and does not depend on frequency.	Velocity. of transverse waves in a bar depends upon frequency.
4.	Medium is nondispersive for longitudinal waves in a bar.	Medium is dispersive for transverse waves in a bar.
5.	Group velocity for longitudinal waves in a bar is equal to wave velocity.	Group velocity for transverse waves in a bar is double of wave velocity.
6.	Overtones are harmonics of fundamental frequency.	Overtones are not harmonics of the fundamental frequency.

16. Tuning Fork

Tuning forks are very important instruments, used as sources of sound of fixed frequency in physical laboratories. They are also used as standards of measuring small time intervals with high accuracy because due to the fixed frequency of the fork, its period of vibration is very precise.

A *tuning fork may be considered as a modified form of a free-free bar, executing transverse vibrations. When such a bar is bent in the shape of big* U *and added with a stem at the centre, we obtain a* **tuning fork.**

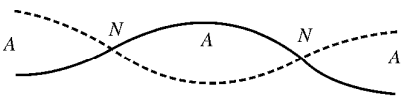


Fig. 11: Transverse vibrations of a free-free bar in dundamental mode.

Let us consider the transverse vibrations of a free-free bar, vibrating in its fundamental [**fig. 11**]. During the course of vibration, the two free ends of the bar move simultaneously along the same direction. Two nodes (*N, N*) are formed close to the ends at a distance 0.2242*l* from either of the free ends and an antinode (*A*) is obtained in between the two nodes. As the bar is bent, the two nodes (*N, N*), are shifted closer and closer [**fig. 12**]. When it takes the final shape of U, the two arms become parallel and in this shape the two nodes (*N, N*) are closest. Consequently, the amplitude of vibration at the middle antinode becomes very small compared to that at the ends. Now the original mode of vibration is the same (fundamental) and the two arms are seen to move inwards or outwards simultaneously. *When a stem is attached at the centre, the system is known to be* **tuning fork.** Since the stem has the mass, its addition to the U bar further reduces the amplitude of vibration of the middle antinode and the two nodes (*N, N*) approach still closer. The outward motion of the arms raises the stem, and their inward motion lowers the stem [**fig. 13**]. In consequence, transverse vibrations of the two arms (prongs of the fork) produces the longitudinal vibrations of the stem. If the stem of the fork is pressed on a surface (resonance box or table), the vibrations of the stem cause small transverse vibrations of the surface.

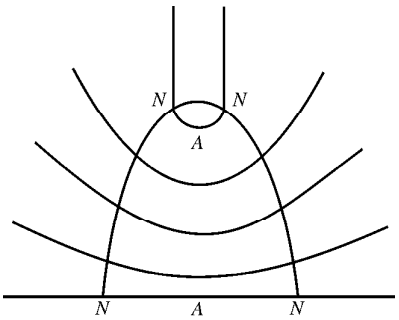


Fig. 12: Bending of a bar to form a tuming fork.

When a tuning fork is struck mildly, it emits fundamental tone only. Each prong of the fork may be considered as a bar fixed at one end and free at the other end (fixed-free bar). The frequency in the fundamental mode of vibration is given by

$$\nu = \frac{(1.88)^2}{l^2} \cdot \frac{K}{2\pi} \sqrt{\frac{Y}{\rho}} \qquad ...(1)$$

We know that for a bar of rectangular cross-section $K = d/\sqrt{12}$, where d is the thickness of the prong. Hence,

$$\nu = \frac{(1.88)^2}{l^2} \cdot \frac{d}{2\pi\sqrt{12}} \sqrt{\frac{Y}{\rho}} \qquad ...(2)$$

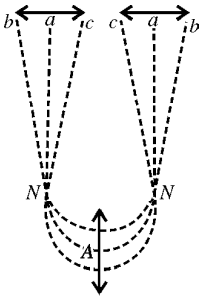


Fig. 13: Vibrating tuning fork.

Thus, the frequency of a tuning fork is

(i) *inversely proportional to the square of its length (l) ,*

(ii) *directly proportional to the thickness of the prong (d),*

and (iii) *directly proportional to the velocity of the longitudinal waves* $\left(v = \sqrt{Y/\rho}\right)$ *in the material of the fork.*

The width of the prong does not affect the frequency of the fork. This is why *the arms of a fork of low pitch are long and thin, whereas the arms of fork of high pitch are small and thick.*

When the fork is struck vigorously overtones, which have frequencies nearly 6 and 17 times that of fundamental, are produced. However these overtones are feeble and die away much more quickly than the fundamental. Thus we obtain only the *pure fundamental tone* which persists for a long time.

Effect of Temperature: When the temperature increases, the length of the fork and thickness of the prong are increased and the density and Young's modulus of the material are decreased. If α be the coefficient of linear expansion of the material of the fork and β the coefficient of decrease of Young's modulus, then the frequency at temperature t is given by

$$\nu_t = \frac{(1.88)^2}{l_0^2(1+\alpha t)^2} \cdot \frac{d_0(1+\alpha t)}{2\pi\sqrt{12}} \sqrt{\frac{Y_0(1-\beta t)}{\rho_0(1-3\alpha t)}}$$

$$= \frac{(1.88)^2}{l_0^2} \cdot \frac{d_0}{2\pi\sqrt{12}} \sqrt{\frac{Y_0}{\rho_0}} \cdot \left(1+\frac{1}{2}\alpha t\right) \left(1-\frac{1}{2}\beta t\right)$$

i.e., $$\nu_t = \nu_0 \left(1-\frac{\beta-\alpha}{2}t\right) \quad ...(3)$$

where ν_0 is the fundamental frequency at 0ºC.

For steel, $\alpha = 12 \times 10^{-6}$ K^{-1} and $\beta = 240 \times 10^{-6}$ K^{-1}

$\therefore$ $$\nu_t = \nu_0 (1 - 0.00011t) \quad ...(4)$$

Thus, *with the rise of temperature the frequency of a tuning fork decreases slowly.* This decrease in frequency is about 0.01% for 1ºC rise in temperature for steel. Now a days, standard forks (standard of frequency) are usually made of elinvar for which the temperature coefficient is very small and the frequency remains approximately constant.

Ex. 1. *What is the wavelength of longitudinal waves of frequency* 400 Hz *in an alloy whose density is* 5,500 kg/m^3 *and Young's modulus* 8.8×10^{10} n/m^2.

Sol. The velocity of sound $$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{8.8\times10^{10}}{5500}} = 4 \times 10^3 \text{ m/sec}$$

$\therefore$ Wavelength $$\lambda = v/\nu = 4 \times 10^3/400 = \mathbf{10\ m.}$$

Ex. 2. *A brass rod of length* 3 *metres is clamped at the centre. It emits a note of frequency* 600 Hz, *when it vibrates longitudinally in fundamental mode. If the density of the brass is* 8300 kg/m^3, *calculate its Young's modulus.*

Sol. When a rod of length clamped in the middle, vibrates in fundamental mode, the expression for the frequency of vibration is

$$\nu = \frac{1}{2l}\sqrt{\frac{Y}{\rho}} \quad ...(i)$$

where Y is Young's modulus and ρ is the density of the material of the rod.

From (i) $$Y = 4\nu^2 l^2 \rho \quad ...(ii)$$

Here, ν = 600 Hz, l = 3 m and ρ = 8300 kg/m^3. Hence

$$Y = 4 \times (600)^2 \times (3)^2 \times 8300 = \mathbf{10.76 \times 10^{10}\ n/m^2.}$$

Ex. 3. *A brass rod* 2m *in length is clamped at its middle point. One end is excited to produce longitudinal vibrations. The frequency of the emitted note is found to be* 1000 Hz. *Calculate the velocity of longitudinal waves and also Young's modulus of brass assuming the density to be* 8500 kg/m^3.

(O.U. 2009)

Sol. When a rod is clamped at its middle point, the frequency of longitudinal vibrations in the fundamental note is given by

$$\nu = \frac{v}{2l}$$

where v is the velocity of longitudinal waves in the rod and l in the length of the rod.

Therefore, $$v = \nu \times 2l = 1000 \times 2 \times = \mathbf{4000m\ sec.}$$

Also $\nu = \sqrt{\frac{Y}{\rho}}$,

$\therefore$ $Y = \rho \nu^2 = 8500 \times (4000)^2 = 1{\cdot}36 \times 10^{11}$ N/m^2.

Ex. 4. *A brass ord of length 1m is clamped at its mid-point. If is is made to vibrate longitudinally, find the first overtone frequency.* (*Y for brass* = 10×10^{10} N/m^2, *density of brass is* 8.3×10^3 kg/m^3) **(O.U. 2010)**

Sol. Frequency $\nu = \frac{n}{2e}\sqrt{\frac{Y}{\rho}}, n = 1,3,5,.....$

First overtone frequency (n = 3) $\nu_1 = \frac{3}{2e}\sqrt{\frac{Y}{\rho}}$

$$= \frac{3}{2\times 1}\sqrt{\frac{10\times 10^{10}}{8.3\times 10^3}} = \frac{3\times 10^4}{\sqrt[2]{8.3}} = 5.21\times 10^3 \text{Hz}$$

Ex. 5. *Calculate the approximate frequency of the lowest note given by the longitudinal vibrations of a steel bar of length* 25 cm., *density* 8000 km/m^3 *and Young's modolus* 2×10^{11} N/m^2.

Sol. The frequency of the lowest note given by the longitudinal vibrations of a free-free bar is given by

$$\nu = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$$

Here, l = 0.25 m, $Y = 2 \times 10^{11}$ N/m^2 and ρ = 8000 kg/m^3

$\therefore$ $\nu = \frac{1}{2\times 0.25}\sqrt{\frac{2\times 10^{11}}{8000}}$ = **10^4 Hz**.

Ex. 6. *A standing wave, produced in a rod, is represented by the equation*

$$y = 0.001 \sin (13x) \sin (200t).$$

If the value of Young's modulus of the material of the rod is 10^{11} N/m^2, *calculate the maximum particle velocity and tensile stress at* x = 4 cm.

Sol. Particle velocity, $dy/dt = 0.001 \times 200 \sin (13x) \cos (200t)$

When $\cos (200t) = 1$, the particle velocity will be maximum.

$\therefore$ Maximum particle velocity = $0.001 \times 200 \sin (13x)$

But at x = 0.04 m, $13x = 13 \times 0.04 = 0.52$ radian = $0{\cdot}52 \times 180/3.14$ degree = 30° approx.

$\therefore$ $\sin (13x) = \sin 30° = \frac{1}{2}$

Hence, Maximum particle velocity = $0.001 \times 200 \times \frac{1}{2}$ = **0.1 m/sec.**

$$\text{Tensle stress} = Y\frac{dy}{dx} = 10^{11} \times 13 \cos (13x) \sin (200t)$$

Hence at x = 0.04, maximum tensil stress

$= 10^{11} \times 0.001 \times 13 \times \sqrt{3}/2$ [$\because \cos (13x) = \cos 30° = \sqrt{3}/2$]

$=$ **11.26×10^8 N/m^2.**

Ex. 7. *A copper rod of length* 0.5 m *is clamped at one end and kept free at the other end. If it is made to execute longitudinal vibrations, find the fundamental frequency and frequencies of the first two overtones. (Young's modulus of copper* = 12×10^{10} Nm^{-2}; *Density of copper* = 8900 kg m^{-3}) **(A.N.U. 2013; O.U. 2014; S.V.U. 2009)**

Sol. When a rod is clamped at one end and kept free at the other end, the frequency of longitudinal vibrations of the rod in the *n*th mode of vibration is given by

$$\nu_n = \frac{(2n-1)}{4l}\sqrt{\frac{Y}{\rho}}, \text{ where } n = 1, 2, 3, \text{ ...etc.}$$

Fundamental frequency of the rod corresponds to $n = 1$ i.e.,

$$\nu_1 = \sqrt{\frac{Y}{\rho}} = \frac{1}{4\times 0.5}\cdot\sqrt{\frac{12\times 10^{10}}{8900}} = \mathbf{1836\ Hz.}$$

For $n = 2$, we obtain the frequency of first overtone i.e.,

$$\nu_2 = \frac{3}{4l}\sqrt{\frac{Y}{\rho}} = 3\nu_1 = 3 \times 1836 = \mathbf{5508\ Hz.}$$

The frequency of second overtone corresponds to $n = 3$ i.e.,

$$\nu_3 = \frac{5}{4l}\sqrt{\frac{Y}{\rho}} = 5\nu_1 = 5 \times 1836 = \mathbf{9180\ Hz.}$$

Ex. 8. *The second harmonic of the note emitted from a bar, free at both ends, has the same frequency as the frequency of the third harmonic note emitted from another bar, clamped at one end and free at the other. Both the bars are executing longitudinal vibrations. Determine the ratio of the length of the two bars. Assume that the two rods are made of the same material.*

Sol. For a free-free bar, frequency of the second harmonic is given by

$$\nu_2 = \frac{2}{2l_1}\sqrt{\frac{Y}{\rho}} \qquad ...(i)$$

From a clamped-free bar, the frequency of the third harmonic

$$\nu_3 = \frac{3}{4l_2}\sqrt{\frac{Y}{\rho}} \qquad ...(ii)$$

According to the question $\nu_2 = \nu_3$ i.e.,

$$\frac{2}{2l_1}\sqrt{\frac{Y}{\rho}} = \frac{3}{4l_2}\sqrt{\frac{Y}{\rho}} \text{ or } \frac{l_1}{l_2} = \mathbf{\frac{4}{3}}.$$

Ex. 9. *A copper rod of length* 2 metre *is free at its ends. The diameter of cross-section of the rod is* 0.01 m. *Find the fundamental frequency of the longitudinal vibrations and transverse vibrations of the rod. (Velocity of sound is copper is* 3560 m/s.) **(M.G.U. 2014)**

Sol. (i) For longitudinal vibrations, the fundamental frequency is given by

$$\nu_t = \frac{\nu}{2l} = \frac{3560}{2\times 4} = \mathbf{445\ Hz}$$

(ii) For transverse vibration, the fundamental frequency is given by

$$\nu_t = \frac{(3.73)^2}{l^2}\cdot\frac{Kv}{2\pi}$$

Here, $K = \frac{r}{2} = \frac{0.01}{2\times 2} = 0.0025$ m, $v = 3560$ m/s and $l = 2$ m

$\therefore$ $$\nu_t = \frac{(4.43)^2\times 0.0025\times 3560}{2^2\times 2\times 3.14} = (4.73)^2\times\frac{8.9}{6.14} = \mathbf{7.925\ Hz}$$

Ex. 10. *A metallic bar of length* 1 metre *is free at its both ends. The diameter of cross section of the rod is* 1 cm. *Find the fundamental frequency of the longitudinal vibrations and transverse vibrations of the rod. (Velocity of sound in copper is* 4000 m/s.)

Sol. When a free-free bar vibrates longitudinally the fundamental frequency of vibration is given by

$$\nu_l = \frac{v}{2l} = \frac{4000}{2\times 1} = \textbf{2000 Hz.}$$

In case of transverse vibrations of a free-free bar, the fundamental frequency of vibration is given by

$$\nu_t = \frac{(4.73)^2}{l^2}\cdot\frac{k}{2\pi} \qquad \left[v = \sqrt{\frac{Y}{\rho}}\right]$$

Here, $K = \frac{r}{2} = \frac{0.01}{2\times 2} = 0.0025$ m, $v = 4000$ m/sec, $l = 1$ m

$\therefore$ $$\nu_t = \frac{(4.73)^2 \times 0.0025 \times 4000}{1^2 \times 2 \times 3.14} = \textbf{35.62 Hz.}$$

QUESTIONS

Long Answer Questions

1. What are longitudinal waves? obtain wave equation and its solution for longitudinal vibrations in a bar. **(Krishna U. 2014)**
2. What are longitudinal vibrations in a bar? Deduce an expression for the velocity of longitudinal waves in a solid bar. **(A.U. 2014)**
3. Set up the differential wave equation when the bar is executing longitudinal vibrations. Give the solution of the longitudinal wave equation.
4. Deduce the wave equation for longitudinal vibrations in a bar in differential form and show that the velocity of the longitudinal waves in the bar is given by
$$\nu = \sqrt{Y/\rho}\,.$$
5. Discuss the boundary conditions used in the theoretical treatment of stationary waves in a bar. Using the solution of longitudinal wave equation, explain the modes of vibration of a bar clamped in the middle.
6. Discuss briefly the longitudinal modes of bibration of a bar damped at the centre and free at the ends. **(T.U. 2014)**
7. Derive the general solution for a longitudinal wave in a bar. Discuss the modes of vibrations for (a) the bar free at both ends (b) the bar fixed at one end. **(S.K.D.U. 2013; S.K.U. 2014)**
8. What are longitudinal waves? Discuss the modes of vibrations of bar clamped at both ends. **(V.S.U. 2014)**
9. Show that the longitudinal vibrations of a free-free bar contain all harmonics while those of a clamped-free bar contain only odd harmonics.
10. Obtain wave equation and its solution for longitudinal waves in a bar, when the bar is:
(i) fixed at both ends, and (ii) free at both ends.
11. Deduce an expression for the fundamental frequency of the longitudinal vibrations of a bar (i) clamped at one end, and (ii) clamped at both ends.
12. Obtain the wave equation for longitudinal vibrations in a bar fixed at both ends and discuss its solution. **(K.U. 2012)**
13. Discuss the stationary waves and first three modes of longitudinal vibrations of a free-free bar.
14. Deduce the differential equation for the transverse waves in a thin rod. Hence show that the rod behaves as a dispersive medium.
15. Assuming a plane wave solution of differential wave equation for the transverse vibrations in a bar, show that the velocity of transverse waves is given by

$$u = \sqrt{\omega v K}$$

where ω is the angular frequency of the wave, v the velocity of longitudinal waves and K the geometrical radius of gyration.

16. Assuming a plane wave solution of transverse wave equation in the form $y = a\, e^{i(\omega t - kx)}$, show that the wave velocity (u) is a function of angular frequency (ω) i.e., the medium is dispersive. Hence show that the group velocity is given by

$$u_g = 2Kk\sqrt{\frac{Y}{\rho}}$$

where symbols have usual meaning.

17. What are transverse and longitudinal waves? Obtain the equation of motion for transverse waves in a bar. **(M.G.U. 2014)**

18. Solve the differential equation in general for simple harmonic transverse vibrations of a bar and discuss the case for free-free bar.

19. Derive a general solution for the transverse wave in a bar. Compare the longitudinal and transverse vibrations. **(A.N.U. 2012)**

20. Discuss the propagation of transverse vibrations in bar and derive wae equation for transverse vibrations of a bar. **(R.U. 2014)**

21. Deduce the expression for normal mode frequencies for transverse vibrations of a bar, clamped at one end and free at the other.

Hence show that the overtones (partials) do not form a harmonic series.

22. For the vibrations of a clamped-free bar in fundamental mode, show that the frequency of transverse vibrations is much less than the frequency of longitudinal vibrations.

23. Explain the vibrations of a tuning fork. Mention the different factors affecting its frequency.

24. Describe the changes in the position of the nodes which occur, when a bar is bent more and more from the straight form to a U-shaped one.

25. Discuss vibrations of a tuning fork. How does the rise of temperature affect the frequency of a tuning fork ?

Short Answer Questions

1. What types of vibrations are possible in a bar? **(Dr. B.R.A.U. 2014; M.G.U. 2014)**

2. Obtain the general solution of longitudinal vibrations in a bar.

3. Obtain the general solution of longitudinal wave equation. **(S.V.U. 2014)**

4. Show that the expression for tensile stress during longitudinal vibrations of a bar is given by

$$F = Y\, dy/dx$$

where terms have usual meaning.

5. Deduce an expression for the velocity of longitudinal sound waves in a bar.

6. Write the equations of standing waves for first three modes of longitudinal vibrations of a free-free bar.

7. Discuss briefly the longitudinal modes of vibration of a bar clamped rigidly at both the ends. **(O.U. 2009)**

8. Show that only odd harmonics are possible for the longitudinal vibrations of a bar clamped at mid point.

9. Set up the differential wave equation for transverse vibrations of a bar.

10. Deduce an expression for the velocity of transverse waves in a bar.

11. Show that the partials in case of transverse vibrations of a bar do not form a harmonic series.

12. Write the boundary conditions for transverse vibrations of (i) a free-free bar, and (ii) fixed-free bar.

13. What are boundary conditions when a bar vibrates (a) longitudinally and (b) transversely. **(S.K.U. 1990)**

14. Do transverse wave travel along the bar with constant velocity? Explain. **(K.U. 2013)**

15. Derive the equation of transvese wave in a bar. **(Dr. B.R.A.U. 2013)**

16. Compare the transverse vibrations of a bar and those of a string.
17. Compare the longitudinal and transverse vibrations in a bar.
18. How can you say that the vibrations of a tuning fork can be regarded as the transverse vibrations of a free-free bar, curved at the middle.
19. What is the expression for the frequency of a tuning fork? Discuss the various factors affecting its frequency.
20. Describe the constructin and working of tuning fork. **(S.V.U. 2014; K.U. 2014; K.U. 2012)**
21. Write a short note on turning forks. **(A.N.U. 2013)**
22. How does the temperature affect the frequency of a tuning fork?

PROBLEMS

1. A metallic rod of length 3 m is clamped at the centre. It emits a note of frequency 600 Hz, when it vibrates longitudinally. If the density of the material of the rod is 8300 kg/m^3, find the value of Young's modulus.
[**Ans.** 1.076×10^{11} N/m^2]

2. The density of aluminium is 2.8×10^3 kg/m^3 and its Young's modulus is 7×10^{10} N/m^2. If the frequency of the aluminium rod is 500 Hz, find the velocity of sound and wavelength through the rod.
[**Ans.** 5000 m/sec, 10m] **(S.V.U. 2011; A.N.U. 2009)**

3. A metallic rod of length one metre and density 7100 kg/m^3 is clamped at its middle and longitudinal vibrations are set up in it by stroking along its length. If the frequency of the fundamental note produced by it is 2600 Hz, find the velocity of sound in the rod and the Young's modulus of the mateial of the rod.
[**Ans.** v = 5200 m/s and $Y = 21.39 \times 10^{10}$ N/m^2]

4. A brass rod, one metre long, is firmly clamped in the middle and one end is stroked by a resined cloth. What is the pitch of the note which you will hear? (Young's modulus for brass = 10^{11} N/m^2 and density = 9,000 kg/m^3)
[**Ans.** 1666.6 Hz]

5. A copper rod of length 1.2 m and diameter 1 mm is clamped at one end. Calculate the frequencies of longitudinal vibrations of the first three harmonics. (For copper, Young's modulus = 12.2×10^{10} N/m^2 and density = 8900 kg m^{-3}).
[**Ans.** 771.3 Hz, 2314 Hz and 3856 Hz]

6. A certain wave produced in a certain medium is expressed as

$$y = 0.002 \sin (0.15x) \sin (48000\ t).$$

where x and y are in cm and *in sec*. Calculate the maximum particle velocity and tensile stress at the point x = 3·5 cm, if Y of the medium = 8×10^{10} N/m^2.
[**Ans.** 0.48 m/sec; $2{\cdot}1 \times 10^7$ N/m^2]

7. A brass rod of length 1 m is clamped at its mid point. If it is made to vibrate longitudinally, find the fundamental frequency and frequencies of the first two overtones. (Y for brass = 10×10^{10} N m^{-2} and density of brass = 8.3×10^3 kg m^{-3})
[**Ans.** 1735.5 Hz; 5206.5 Hz and 8677.5 Hz]

8. A wooden rod of length 1 m, clamped at its centre, is stroked with a resined cloth. A note of frequency 2048 Hz is heard. Find Young's modulus for the material of rod. (Density of wood = 800 kg/m^3)
[**Ans.** 1.34×10^{10} N/m^2]

9. In the apparatus of Kundt's tube, a copper rod of one metre length is clamped at a distance of 25 cm from one end. Air is present in the tube in which the velocity of sound is 340 m/sec. Powder in the tube is collected in small heaps with 5 cm distance between two in succession. What is velocity of sound in copper?
[**Ans.** 3400 m/sec]

10. A steel rod is of 0.002 m radius and has a length of 2 m. What is its fundamental frequency of free-free transverse vibrations? (Young's modulus $Y = 19.5 \times 10^{10}$ N/m^2 and density $\rho = 7700$ km^{-3})
[**Ans.** 4.47 Hz]

11. A copper rod of length 2 m is free at its both ends. The diameter of the rod is 1 cm. Find the fundamental frequency of longitudinal and transverse vibrations of the rod. (Velocity of sound in copper = 3560 m/sec.)
[**Ans.** 890 Hz; 7.9 Hz]

12. An aluminium bar, 1.2 m long, has a rectangular cross-section with a thickness of 0.006 m, measured in the direction of transverse vibrations. Calculate the fundamental frequency and two overtones of transverse vibrations of the fixed-free bar. (Velocity of longitudinal waves in the bar = 5150 m/s).
[**Ans.** 3.48 Hz, 21.82 Hz, 61.07 Hz]

[**Hint.** $\nu = \frac{(1.88)^2}{l^2} \cdot \frac{Kv}{2\pi} = \frac{(1.88)^2 \times 0.006 \times 5150}{(1.2)^2 \times 2\sqrt{3} \times 2 \times 3.14} = \frac{109.2}{31.3} = 3.48$ Hz. $\left[\because K = \frac{d}{2\sqrt{3}}\right]$]

13. An iron rod of length 2 m, fixed at the mid point, is struck so that it vibrates longitudinally. The fundamental note emitted by the rod is in resonance with the sonometer wire of length 16.2 cm. Keeping the tension in the wire unchanged, a tuning fork of frequency 540 Hz is found in resonance with the same sonometer wire of length 72.6 cm. Calculate the frequency of the iron rod and its Young's modulus. (Density of steel 7700 kg/m^3)
[**Ans.** 2420 Hz, 1.8×10^{11} N/m^2]

14. A wooden rod of length 1.5 m is held in the middle and set in vibration longitudinally. It gives the same note as an open organ pipe 1.3 m long. Find the speed of sound in the rod. (Speed of sound in air = 338 m/sec.)
[**Ans.** 388.7 m/sec.]
[**Hint.** For an open organ pipe, frequency of fundamental $\nu = v/2l$]

●

6

Ultrasonics

1. Introduction

Sound waves can be broadly classified into three classes on the basis of their frequencies:

(1) Sonics or Audible Waves: The sound waves whose frequencies are in the range 20 Hz to 20 kHz and which can be heard by human ear are called **sonics** or **audible waves**. These waves are produced by the vibrating bodies, e.g., by stretched strings-violin, guitar, vocal cords - animal and human voice, vibrating membranes and plates - drum, loudspeaker and vibrating air columns - clarinet, organ pipe.

(2) Subsonic or Infrasonic Waves: The sound waves, which have frequencies less than the lower limit of the audible range i.e., 20 Hz, are called subsonic or infrasonic waves. Our ear does not respond to these waves. Such waves are produced by earthquake and vibrating sources of low frequencies (< 20 Hz).

(3) Ultrasonic Waves: The sound waves, which have frequencies greater than the upper limit of audible range i.e., 20 kHz, are called ultrasonics or ultrasonic waves. As the frequencies of the ultrasonic waves are very high, these waves do not stimulate our ear and are not heard. However some animals like bats and dogs respond to the ultrasonic waves. The wavelength of these waves in air is of the order of 1 cm or less ($\lambda = v/\nu = 340/20{,}000$ m = 1.7 cm). At present, ultrasonic waves of frequencies of the order of 10^9 Hz are conveniently produced.

Hypersonic Waves: The frequency of these waves is more than 5×10^8 Hz.

Supersonic Speed: A body moving with a speed higher than the speed of sound is said to be moving with **supersonic speed**.

Shock Waves: Shock waves are produced when a sudden and abnormal increase of pressure and density occurs at any place in the medium. For example, when a jet aircraft flies in the atmosphere at supersonic speeds (i.e., at speeds greater than the speed of sound), a strong shock wave is generated and radiated from the aircraft so that a loud booming sound is heard. If the speed of the aircraft or any body is v and v_s in the speed of sound, then the ratio v/v_s is called **Mach number**. If the mach number exceeds 1, the body is moving at a supersonic speed.

Because of several important applications in the fields of metallurgy, medical, biology etc., ultrasonics need to be discussed specially. In the present chapter, we shall deal with the production, detection, measurement and applications of these waves.

2. Production of Ultrasonic Waves

Generally ultrasonic waves are produced by two methods: Magnetostriction method and Piezo-electric method.

(1) Magnetostriction Method: Ultrasonic waves can be generated by using the principle of magnetostriction. If a rod of ferromagnetic material (e.g., iron, nickel) is placed along an alternating magnetic field, its length is periodically increased and decreased. This phenomenon is called **magnetostriction effect.** If the natural frequency of longitudinal vibrations of the rod is equal to that of the alternating current producing the alternating magnetic field, a resonance is obtained and the amplitude of oscillations of the rod becomes maximum. Ultrasonic waves of the same frequency are produced by

the oscillating rod. If the rod has been clamped in the middle, then this is the case of longitudinal vibrations of a rod fixed at the middle and free at both ends. The frequency of longitudinal vibrations of such a rod in the fundamental mode of vibration is given by

$$\nu = \frac{v}{\lambda} = \frac{v}{2l} \qquad [\because \lambda = 2l] \quad ...(1)$$

where v is the velocity of ultrasonic waves and l is the length of the rod.

The velocity of ultrasonic waves (sound) in a rod, vibrating longitudinally, is given by

$$v = \sqrt{\frac{Y}{\rho}} \qquad ...(2)$$

where Y is the Young's modulus and ρ the density of the material of the bar.

Hence the frequency in the fundamental mode is obtained to be

$$\nu = \frac{1}{2l}\sqrt{\frac{Y}{\rho}} \qquad ...(3)$$

If the rod is vibrating longitudinally in the nth harmonic, then

$$\nu = \frac{n}{2l}\sqrt{\frac{Y}{\rho}} \qquad ...(4)$$

where $\nu = 1, 3, 5...$

The high frequency alternating current required to generate the ultrasonic waves is obtained from an oscillatory circuit. Here, the iron or nickel rod serves as a *transducer* to convert the electrical energy into sound waves.

Working of Magnetostriction Vibrator: In the magnetostriction method, *AB* a rod of nickel, cobalt or iron, is clamped at the middle. L_1 and L_2 are two coils surrounding the *AB* rod. The diameter of the turns of the coil is greater than that of rod, thus leaving the rod free to move lengthwise, inside the coils [**fig. 1**]. L_1 is forming a tuned circuit with the variable condenser *C* which is connected to the plate *P* of a triode valve. The coil L_2 is connected between the grid *G* and the cathode *F*. The values of L_1 and *C* give the frequency of the oscillatory circuit. If this frequency of the *LC* circuit is equal to the frequency of the rod, then the longitudinal oscillations are maintained and ultrasonics are produced in the surrounding medium.

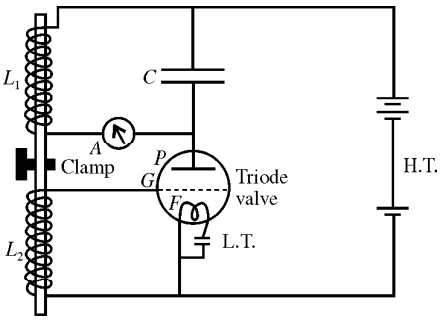


Fig. 1: Magnetostriction vibrator.

In the beginning, the bar is magnetised by passing direct current. The periodic variation in the anode current, passing in the coil L_1, causes a variation in magnetisation and resulting in a variation in the length of the rod. This variation in length causes a change in the magnetic flux through the grid coil L_2. Hence, by converse magnetostriction effect, an induced emf is produced in L_2. This emf acts on the grid and causes an amplified current variation in L_1. Thus the oscillations of the rod are maintained.

Initially the milliammeter *A* shows alteration when the oscillations are set up. The capacitor *C* is adjusted so that these alterations are maximum. In this position, frequency of the oscillating anode current coincides with the natural frequency of the rod. Now the vibrations are most intense and ultrasonic waves are produced.

Ultrasonics of frequencies in the range 5 to 60 kHz are produced by the magnetostrictive vibrator. The ultrasonic waves of high frequencies are produced by making use of Piezo-electric effect.

Transistorised Circuit of Magnetostrictive Oscillator

The circuit of magnetostrictive oscillator, which is the transistorised form of the original circuit developed by Pierce, is shown in **fig. 2**. Here the ends of a rod *AB* of ferromagnetic material is wound by the coils L_1 and L_2. The tuned circuit L_1C_1 is connected to the collector of the transistor *T*. The other

end of the rod carries coil L_2, which is connected to the base of the transistor as shown in **fig. 2**. The frequency of the oscillatory circuit (L_1C_1) can be adjusted by the variable condenser C_1. The current can be measured by the milliammeter connected across the coil L_1. The battery connected between the emitter and the collector provides required biasing *i.e.*, the emitter is forward biased and the collector is reverse biased for *NPN* transistor. Therefore, the current can be produced by applying necessary biasing to the transistor by using the battery.

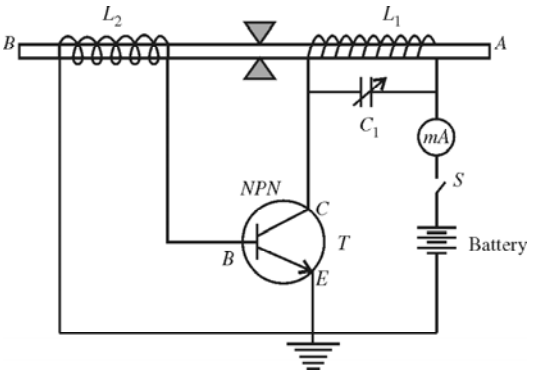


Fig. 2.

In the beginning the rod is permanently magnetized by passing a d.c. current. By switching on the battery, the current is produced in the transistor. The current is passed through the coil L_1. The resulting magnetic field changes the length of the rod due to magnetostriction. This changes the magnetic flux through the coil L_2 and hence induces on *e.m.f.* in it due to converse magnetostriction effect. The induced *e.m.f.* is fed in the base of the transistor and it enhances the forward bias of the base emitter juction. Hence the collector cuttent increases. Thus a positive feedback, needed to sustain the oscillator is provided.

The frequency of the oscillatory circuit is adjusted by variable condenser and when this frequency is equal to the frequency of the oscillating rod, resonance occurs. At resonance, the rod oscillates longitudinally with larger amplitude and ultrasonic waves of high frequency are produced along both ends of the rod. The condition of resonance is the frequency of the oscillatory circuit is equal to the frequency of the oscillating rod (say in fundamental mode) *i.e.*,

$$\frac{1}{2\pi\sqrt{LC}} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$$

Advantages

(i) Forromagnetic materials in magnetostriction method are easily available and have low cost.

(ii) Large output power can be obtained in this method.

Disadvantages

(i) In magnitostricticton method, frequencies can be produced upto 3 MHz only.

(ii) We know that the frequency is inversely proportional to the length of the oscillating rod. Therefore to increase the frequency, the length of the rod is to be dicreased which is practically not possible.

(iii) From this oscillator it is not possible to get a constant frequency, as the rod depends on temperature and the degree of magnetization.

(2) Piezo-electric Method: If a plate, cut in a special way from a crystal of quartz, tourmaline, rockelle salt etc., is compressed, the other faces of the plate acquire opposite electric charges. When the plate is stretched, the signs of the charges are changed. This phenomenon is called **Piezo-electric effect.** Converse of Piezo-electric effect is also true. Thus, if the plate is placed in an alternating electric field, it contracts and expands periodically resulting in the mechanical vibrations. The frequency of vibrations of the crystal plate is in the ultrasonic range.

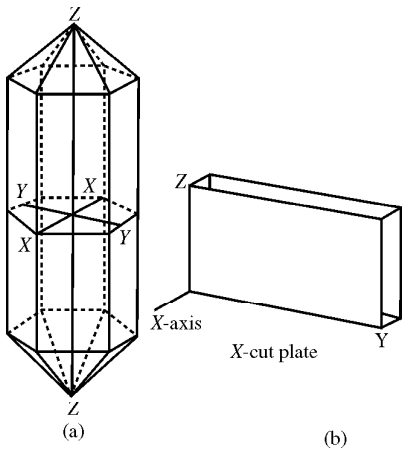


Fig. 3: Shape of quartz crystal plate.

Shape of the Crystal Plate: In the Piezo-electric vibrator, generally quartz crystal in the form of hexagonal pyramid, is used to obtain the crystal plate. The line ZZ along the long side of the crystal is called **optic axis** or **Z-axis.** *XX* line, which is perpendicular to *Z*-axis and connecting any two opposite corners of the hexagon, is called **electric axis** or *X*-**axis** [**fig. 3(a)**]. They are three in number. Now, if a line *YY* is drawn in such a way that it is perpendicular to both *X*- and Z- axes and normal to the two opposite faces, this line is called **mechanic axis** or *Y*-**axis**. They are also three in number. When a plate is cut from a crystal, so that both of its principal faces are perpendicular to *X*-axis, the plate is called *X*-**cut crystal**. Similarly, if the plate is cut from a crystal having both of its faces perpendicular to *Y*-axis, the plate is called *Y*-**cut crystal** [**fig. 3(b)**]. The distance between the two principal faces is the thickness of the plate in case of *X*-cut or *Y*-cut crystal [**fig. 4**].

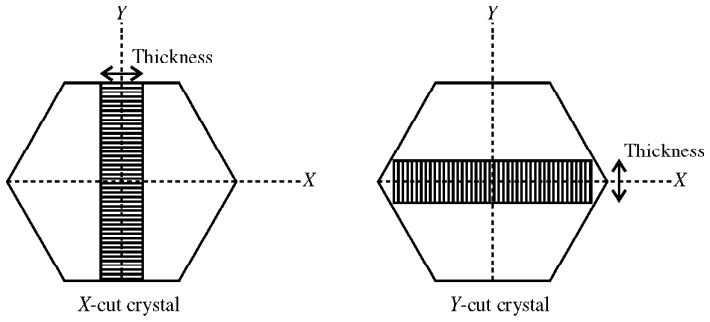


Fig. 4: Transverse section of hexagonal quartz crystal.

Resonant Vibrations of Crystal Plate: To produce ultrasonic waves (longitudinal), *X*-cut crystal is frequently used. When an alternating electric field is applied along *X*-axis of the *X*-cut plate, the plate either expands along the *Y*-axis (length) and contracts along the *X*-axis (thickness) or shows contraction along the *Y*-axis and expansion along the *X*-axis, depending on the direction of the electric field. Thus the crystal plate will execute forced vibrations. If the frequency of the alternating electric field is equal to the frequency of the crystal plate, then the phenomenon of **resonance** takes place and the amplitude of vibration becomes large. The vibrational mode along the *Y*-axis (length) is called the **length vibration** whereas that along the *X*-axis (thickness) is known to be **thickness vibration.** These vibrations can be considered longitudinal, whether they are due to the strain in the length or due to the strain in the thickness. The two possible frequencies in the fundamental mode are given by

$$\nu_l = \frac{v}{\lambda} = \frac{v}{2l} \quad [\because \lambda = 2l] \text{ for length vibration} \qquad ...(1)$$

and

$$\nu_t = \frac{v}{\lambda} = \frac{v}{2t} \quad [\because \lambda = 2t] \text{ for thickness vibration} \qquad ...(2)$$

Since $v = \sqrt{Y/\rho}$ [Y = Young's modulus and ρ = density of the crystal],

$$\nu_l = \frac{1}{2l}\sqrt{\frac{Y}{\rho}} \text{ for length vibration} \qquad ...(3)$$

and

$$\nu_t = \frac{1}{2l}\sqrt{\frac{Y}{\rho}} \text{ for thickness vibration} \qquad ...(4)$$

When the crystal is vibrating in the *n*th mode, the two frequencies are given by

$$\nu_l = \frac{v}{\lambda} = \frac{n}{2l}\sqrt{\frac{Y}{\rho}} \quad [\because \lambda = \frac{2l}{n}] \text{ for length vibration} \qquad ...(5)$$

and

$$\nu_t = \frac{v}{\lambda} = \frac{n}{2t}\sqrt{\frac{Y}{\rho}} \quad [\because \lambda = \frac{2t}{n}] \text{ for thickness vibration} \qquad ...(6)$$

where n = 1, 2, 3, ...

For quartz crystal, if the average value of Young's modulus Y is 7.7×10^{10} N/m^2 and ρ = 2650 kg/m^3, then $\sqrt{Y/\rho} = \sqrt{7{\cdot}7\times10^{10}/2650} = 5.39 \times 10^3$ m/sec. Hence if l = 0.05 m and t = 0.005 m, the fundamental frequencies of the plate are

$$\nu_l = \frac{1}{2\times 0.005}\times 5.39\times 10^3 = 5.39 \times 10^4 \text{ Hz}$$

and $$\nu_t = \frac{1}{2\times 0.005}\times 5.39\times 10^3 = 5.39 \times 10^5 \text{ Hz}.$$

Thus, the length vibration produces low frequency and the thickness vibration generates high frequency. The thickness vibration generally used to generate longitudinal ultrasonic waves in the medium *because of much greater radiating surface.* The frequencies of other modes of the plate are an integral multiple of the fundamental frequency. Hence, if a *X*-cut quartz plate in thickness is vibrated by an alternating electric field of a proper frequency, longitudinal ultrasonic waves are produced. Piezo-electric vibrator can produce the frequencies upto the order of 5×10^8 Hz. *Y*-cut quartz plate is mostly used to generate shear waves in the medium.

Working of Piezo-electric Vibrator: In **fig. 5,** a Piezo-electric vibrator has been shown to produce ultrasonic waves. This consists of a *X*-cut quartz plate *Q*, placed in between the metallic plates *A* and *B*. This arrangement, using the crystal plate as a dielectric, constitutes a parallel plate condenser. The plates *A* and *B* are connected to the terminals of a coil L_1. The coil L_2 is connected to the plate circuit of a triode valve. The coil *L* with capacitor *C* costitutes the oscillating tank circuit and is connected between the grid *G* and filament (cathode) *F*. When the frequency of oscillations of the oscillating circuit is made equal to the natural frequency of the crystal mode, *resonance* occurs and the crystal undergoes linear expansions and contractions at the same rate with relatively large amplitude. In other words, the crystal vibrates vigorously, producing ultrasonic waves in the surrounding medium.

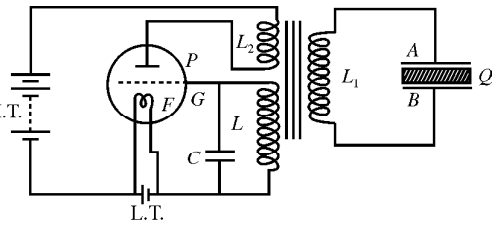


Fig. 5: Piezo-electric vibrator.

Remember that the frequency of the oscillating circuit with inductance *L* and capacitance *C* is given by

$$\nu = \frac{1}{2\pi\sqrt{LC}}.$$

By adjusting the variable capacitance *C* of the tank circuit, the quartz crystal is made to oscillate at its natural frequency to produce resonance.

Tranistorised Cireuit for Piezoelectric Vibrator

Circuit diagram of Piezoelectric vibrator is shown is **fig. 6**. Here a quartz slab *Q* is placed in between of *A* and *B* metallic plates. In the figure, the tank circuit has a variable capacitor C_1 and an inductance L_1 to produce electrical oscillations. The plates *A* and *B* are connected to the terminals of a coil *L*. The coil L_2 is connected between base and emitter of the transistor, say *NPN*. The frequency of electrical oscillations is decided by the tank circuit. The circuit is switched on, current rushes through the tank circuit and the capacitor is charged. When the capacitor is fully charged no current passes through it. Then the capacitor starts to discharge through the inductor and therefore the electric energy is in the form of electric and magnetic fields, associated with the capacitor and the inductor respectively. Thus we obtain electrical oscillations in the tank curcuit and with the help of the transistor, electrical oscillations are produced continuously.

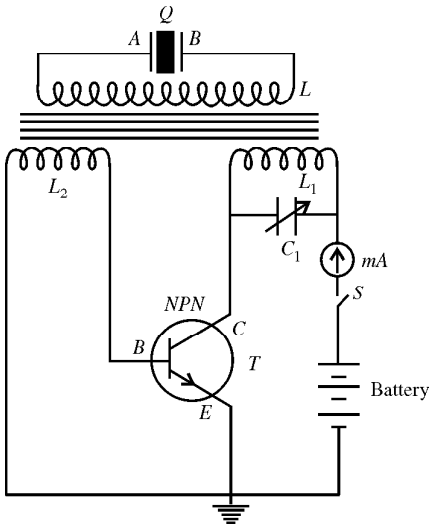


Fig. 6.

This is fed to the secondary circuit and the crystal quartz vibrates, as it is continuously subjected to varying (alternating) electric field and produces ultrasonic waves. When the frequency of oscillations is equal to the natural frequency of the quartz cystal, then it vibrates with maximum amplitude. The frequency generated is given by:

$$\nu = \frac{n}{2l}\sqrt{\frac{Y}{\rho}}$$

where Y is the Young's modulus of the Piezoelectric material and ρ. its density.

Advantanges

(i) Piezoelectric vibrator is more efficient than the magnetostriction oscillator.

(ii) This method can produce frequenceries of the order of 500 MHz.

(iii) In this method the output is very high.

(iv) Piezoelectric vibrator is not affected by termperature and humility.

Disadvantages

(i) In Piezoelectric method, quartz crystal has very high cost.

(ii) The shaping and cutting of the quartz plate is very complex.

3. Detection of Ultrasonics

Few methods to detect ultrasonic waves are given below :

(1) Piezo-electric Detector: Ultrasonics can be detected by using the phenomenon of Piezo-electric effect. When ultrasonic waves are incident along the mechanical axis (*Y*-axis) of a *X*-cut quartz crystal, these waves produce alternating mechanical stresses along the *Y*-axis. This develops small alternating e.m.f. along its electrical axis (*X*-axis). This small emf is amplified and detected by an electronic circuit. Thus the Piezo-electric vibrator with high gain amplifiers and an electronic circuit detects the ultrasonic waves.

(2) Kundt's Tube Method: In a Kundt's tube lycopodium power is spread and then the ultrasonic waves are passed in the tube. It is seen that the powder collects at nodal points. Thus we get $\lambda/2$, being the distance between two nodes. If we know the frequency of the wave, velocity can be calculated by the relation $v = \nu\lambda$. In case of liquid medium, powdered coke is used in place of lycopodium. This method can be used only when the wavelength of ultrasonic wave is not very small i.e., not less than few millimeters.

(3) Thermal Detectors: In this method a thin platinum wire is moved along the direction of propagation of ultrasonic wave. As compressions and rarefactions occur at nodal points, consequently adiabatic change in temperature takes place. Here the resistance of platinum vary with temperature. This variation in temperature can be detected by wheatstone bridge. At the points of antinodes, the temperature and resistance remain constant. This can also be seen by the balanced position of the bridge. This method is very simple and used generally for this purpose.

(4) Sensitive Flame Method: By moving a flame along the medium, we observe that the flame is steady at the positions of antinodes and it flickers at nodal points due to the change of pressure here. Thus the distance between two adjacent nodes ($\lambda/2$) can be found. If we know the frequency of ultrasonic waves, its velocity can be determined.

4. Determination of Wavelength and Velocity of Ultrasonic Waves

(1) Ultrasonic Interferometer Method or Stationary Wave Method: The velocity of ultrasonic waves in a gas or liquid can be determined by acoustic interferometer method. In this method, stationary ultrasonic waves are formed in the gas or liquid which enable us to determine the wavelength (λ) and velocity (v) of the waves. This is why this method is also called ***stationary wave method***.

In the experiment, quartz crystal *Q* is mounted in front of a smooth reflecting surface *R* and the liquid is filled in the tank *T*, as shown in **fig. 7**. The linear dimension of the crystal *Q* are large in comparision to the wavelength of ultrasonic waves and hence almost plane longitudinal waves travel towards the

reflector *R* and are reflected back to the crystal *Q*. The incident and reflected waves form a system of stationary waves. The position of the reflector *R* can be changed by means of a micrometer screw M. When the distance of the reflector from the face of the vibrating crystal is varied, the current in the micro ammeter in the anode circuit of the valve represented by *A*, shows maxima and minima, dependings on the phase of the oncoming waves incident on the crystal. The maxima correspond to the positions of nodal planes and are determined by moving the reflecting surface towards or away the crystal vibrator. The distance between two successive maxima (or minima) gives half the wavelength i.e., $\lambda/2$ **(fig. 8)**. Hence one can know the wavelength λ of the ultrasonic waves, produced in the experiment. The frequency ν of the crystal vibrator is determined by a standard electrical wave meter method. The velocity of ultrasonic waves in the medium (liquid or gas) can then be calculated from the formula:

$$v = \nu \lambda$$

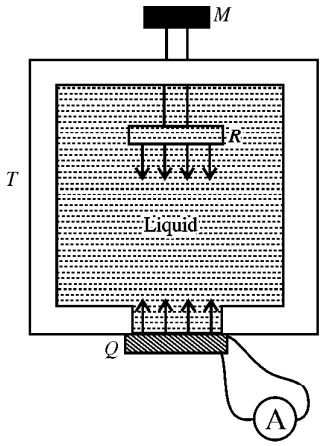


Fig. 7: Experimental set up for the determination of the velocity of ultrasonic waves in a liquid.

(2) Acoustic Grating Method: In order to find the wavelength and hence the velocity of ultrasonic waves, an allternative method known as acoustic grating method, was first suggested by **Brillouin** in 1925. According to the suggestion, when ultrasonic waves are propagated in a liquid, the density varies from layer to layer due to periodic variation of pressure. Now, if monochromatic light is passed through the liquid at right angles to the waves, the liquid acts as a diffraction grating, the lines of compression being lines of greater density. Such a grating is called **acoustic grating.** This grating behaves similar to a ruled grating in optics. In 1932, **Debye and Sears** have used this technique with a stationary wave system of ultrasonics in a cell.

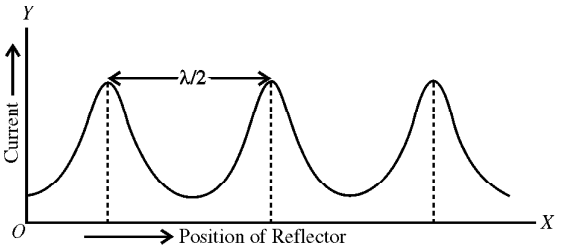


Fig. 8: Variation of the mierommeter current with the position of the reflector.

In the experimental arrangement, a quartz crystal plate *Q* is placed in a glass cell, containing the liquid and is connected to an oscillating circuit, producing plane ultrasonic waves. These waves are reflected by a reflector *R* [**fig. 9**]. The position of the reflector *R* can be adjusted so that the incident and reflected waves produce longitudinal stationary waves in the liquid. In these stationary waves, nodal and antinodal planes are formed at fixed positions. At the nodal planes, the density is maximum due to compression and the antinodal planes, the density is minimum due to rarefaction. These (equally spaced) nodal planes and the inter-nodal spaces in the liquid act like the grating elements. Thus, the liquid with periodic variations in density in space constitutes the acoustic grating.

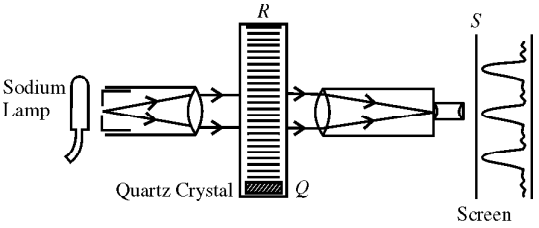


Fig. 9: Acoustic grating.

The acoustic grating is mounted on the prism table of a spectrometer. A parallel beam of sodium light, obtained from the collimator, is incident normally on the acoustic grating *QR* [**fig. 9**]. The light is diffracted by the grating and the diffracted beam of light is seen through the telescope. The diffraction pattern consists of a central maximum and a number of principal maxima on either side of it.

Let θ_n be the angle of diffraction of *n*th principal maximum. If *d* be the distance (grating element) in between two consecutive nodal planes, then the grating equation from physical optics is

$$d \sin \theta_n = n\lambda \quad ...(1)$$

where λ is the wavelength of sodium light.

If λ_a is the wavelength of the ultrasonic waves through the liquid, then

$$d = \frac{\lambda_a}{2} \quad \text{or} \quad \lambda_a = 2d \quad ...(2)$$

Determining d from eq. (1) and substituting it in eq. (2), we can find the **wavelength (λ_a) of the ultrasonic waves.**

Hence the **velocity of ultrasonic waves** in the given medium (liquid) can be obtained as

$$v = \nu\lambda_a \text{ or } v = 2\nu d \quad ...(3)$$

where ν is the frequency of ultrasonic waves.

(3) Pulse Amphitude Method: In this method, a pulse of short duration of ultrasonic waves sent out by the quartz crystal towards a reflector. The reflected pulse received by the crystal is observed on a synchroscope. The time interval t between the emission of the pulse and its reception by the crystal is noted. As the ultrasonic waves travel twice the distance d between the crystal and the reflector in the duration t, hance one can find the velocity of the waves by using the relation

$$v = \frac{2d}{t}$$

5. Applications of Ultrasonic Waves

(1) Underwater Acoustics: Recently underwater acoustics have become an important tool for oceanographers. Because of the extremely short wavelength of the ultrasonic waves, these waves can be obtained in a very narrow beam. Ultrasonic waves of high frequency are used to find the depth of the sea by ***echo-sounding method***. In this method, ultrasonic waves, produced by a Piezo-electric quartz oscillator, are directed towards the bed of sea. These waves are reflected back from the bed of the sea and the echo is detected by the crystal itself. Let t be the time interval between the emitted ultrasonic pulse and its echo. If v be the velocity of sound in the sea water then the depth (d) of the sea is obtained from the reation :

$$d = v \times \frac{t}{2} = \frac{vt}{2}.$$

The echo-sounding method is also used to detect the presence and depth of submarines, sea rocks, icebergs, sunken ships and different objects in ocean. The whole system, used to detect an object inside the sea, is called **sonar**, which means '**sound navigation and ranging**'. By the use of sonar, one ship can converse with the another.

(2) Application in Mining: Ultrasonic techniques are used in mining to get information about rocks of the crust of the earth. The echo-sounding method enables us to detect voids, fractures and ore bodies in the rocks.

(3) Metallugical Applications: The principle of sending out ultrasonic waves and measuring the reflection interval is used in industry to find the location of the flaw inside a metal block or steel bar. If a pulse of ultrasonic waves reflects back from a flaw at a distance d in a time interval t_1 and from the other end of a bar of length l in a time interval t_2, then

$$d = \frac{vt_1}{2} \text{ and } l = \frac{vt_2}{2}$$

where v is the velocity of sound.

Therefore, $$\frac{d}{l} = \frac{t_1}{t_2} \text{ or } d = \frac{lt_1}{t_2}$$

Hence by using echo-sounding method, one can measure the distance of flaw in a steel bar.

(4) Medical Applications: (i) Abnormal growth in the brain, tumours etc. can be detected by ultrasonic waves. They cannot be detected by *X*-rays.

(ii) Ultrasonic message (vibrations) are used to relieve the pain due to arthritis.

(iii) Ultrasonic waves are used in bloodless surgery. Here, ultrasonic waves of high frequency are focussed on the unwanted tissue or part of the body and the tissue is destroyed.

(5) Chemical Applications: (i) High polymers and molecular chains can be broken by high frequency vibrations.

(ii) Metals such as copper-lead, zinc-lead, etc. aluminium-cadmium etc. are alloyed with uniform composition by passing ultrasonic waves of high intensity.

(iii) Ultrasonic waves accelerate the process of crystallization.

(iv) When ultrasonic waves are passed in potassium iodide, iodine is produced.

(6) Heating Effect: When ultrasonic waves are passed in a substance, it is heated. Water can be boiled by these waves.

(7) Uses of Ultrasonics in Nature: (i) The echo-sounding method is used by bats to detect obstacles in their path. Actually bats have natural power of emitting and detecting, ultrasonic waves. During the flight in darkness, they continuously emit these waves and detect echos due to reflection from an obstacle ahead and change their path to avoid the obstacles.

(ii) The dog can hear the sound of more then 20 kHz. Hence a dog whistle, using ultrasonic waves, can be used to call a dog, which will not be heard by the neighbours, without annoying them.

Ex. 1. *A magnetostriction oscillator has a frequency* 20 kHz. *If it produces sound wave of velocity* 6.2×10^3 m/sec, *find the length of the ferrite rod*.**(S.K.U. 2008; O.U. 2009; S.V.U. 2010, 07; Y.V.U. 2014)**

Sol. In a magnetostriction oscillator the frequency of vibration is given by

$$\nu = \frac{v}{2l} \qquad ...(i)$$

where v is the velocity of sound in the rod of length l.

Therefore, $$l = \frac{v}{2\nu} \qquad ...(ii)$$

Here, $v = 6.2 \times 10^3$ m/sec and $\nu = 20 \times 10^3$ Hz.

Hence, $$l = \frac{6.2 \times 10^3}{2 \times 20 \times 10^3} = \mathbf{0.155\ m.}$$

Ex. 2. *A quartz crystal thickness* 0.005 *metre is vibrating at resonance. Calculate the fundamental frequency. Given Y for quartz* $= 7.9 \times 10^{10}$ newton/m^2 *and* ρ *for quartz* $= 2650$ kg/m^3.

(A.N.U. 2010, 05; S.K.U. 2002; O.U. 2014)

Sol. The expression for fundamental frequency of a quartz crystal of thickness t is given by

$$\nu_t = \frac{1}{2t}\sqrt{\frac{Y}{\rho}}$$

Here, $t = 0.005$ m, $Y = 7.9 \times 10^{10}$ N/m^2 and $\rho = 2650$ kg/m^3.

$\therefore$ $$\nu_t = \frac{1}{2 \times 0.005}\sqrt{\frac{7.9 \times 10^{10}}{2650}} = \mathbf{5.46 \times 10^5\ Hz.}$$

Ex. 3. *A Piezoelectric crystal has a thickness* 0.002 m. *If the velocity of sound wave in crystal is* 5750 m/s, *calculate the fundamental frequency of crystal*.

(K.U. 2013; A.U, 2014; A.N.U. 2012; O.U. 2013; M.G.U. 2014)

Sol. Fundamental frequency for the thickness vibration is given by

$$\nu = \frac{v}{2t} = \frac{5750}{2 \times 0.002} = 1.4375 \times 10^6 \text{ Hz} = \mathbf{1.4375\ MHz.}$$

Ex. 4. *A Piezoelectric crystal with vibrating length* 3×10^{-3} m *has density* 3.5×10^3 kg/m^3. *If it is made of a material of Young's modulus* 8×10^{10} N/m^2, *what is its fundamental frequency*?

(A.N.U. 2014, 08; A.U. 2012; S.V.U. 2010)

Sol. The fundamental frequency for length vibration is given by

$$\nu = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$$

Here, $l = 3 \times 10^{-3}$ m, $Y = 8 \times 10^{10}$ N/m^2 and $\rho = 3.5 \times 10^3$ kg/m^3

$$\therefore \quad \nu = \frac{1}{2\times(3\times10^{-3})}\sqrt{\frac{8\times10^{10}}{3.5\times10^3}} = 0.7967 \times 10^6 \text{ Hz} = \mathbf{0.7967\ MHz.}$$

Ex. 5. *The velocity of sound in a Piezoelectric X-cut crystal of thickness* 0.003 m is 5750 ms^{-1}. *Calculate the fundamental frequency of the ultrasonic waves produced.* **(A.U. 2001)**

Sol. Fundamental frequency of *X*-cut crystal is given by

$$\nu = \frac{v}{2t}$$

Here, v = 5750 m/sec and t = 0.003 m

$$\therefore \quad \nu = \frac{5750}{2\times0.003} = 0.958 \times 10^6 \text{ Hz} = \mathbf{0.958\ MHz.}$$

Ex. 6. *Calculate the capacitance to produce ultrasonic waves of* 10^6 Hz *with an inductance of* 1 henry.

(S.V.U. 2011; A.N.U. 2013; K.U. 2002)

Sol. Frequency of *LC* circuit $\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$

whence $$C = \frac{1}{4\pi^2\nu^2 L}$$

Here, L = 1 henry and $\nu = 10^6$ Hz

$$\therefore \quad C = \frac{1}{4\times(3.14)^2\times(10^6)^2\times1} = 0.025 \times 10^{-12} = \mathbf{0.025\ pF.}$$

Ex. 7. *Ultrasonic pulse echo method is employed to detect possible defect in a steel bar of thickness* 40 cm. *If the pulse arrival times are* 30 *and* 80 microseconds, *determine the distance of the defect.*

Sol. Suppose x be the distance of the defect from the end of bar at which the pulse enters. The pulse, arriving first, comes from defect after reflection and the second pulse arriving later on, comes from the other end of the bar after reflection. This means that 30 microseconds are taken to move a distance of $2x$ and 80 microseconds are taken to cover a distance of 80 cm or 0.8 m. Thus

$$30\ \mu\text{s} = \frac{2x}{v} \qquad \text{...(i)}$$

$$80\ \mu\text{s} = \frac{0.8}{v} \qquad \text{...(ii)}$$

Dividing (i) equation by (ii), we get

$$\frac{2x}{v}\times\frac{v}{0.8} = \frac{30}{80}, \ \therefore \ v = \frac{3}{8}\times\frac{0.8}{2} = \mathbf{0.15\ m\ or\ 15\ cm.}$$

Ex. 8. *A crystal is generating ultrasonics of frequency* 552 Hz. *Another crystal of half the thickness and four times Young's modulus is used under the same conditions. What is the frequency of generated ultrasonics?* **(K. U. 2014)**

Sol. Frequency $$v = \frac{1}{2t}\sqrt{\left(\frac{Y}{\rho}\right)}$$

When the crystal is generating frequency v = 552 Hz,

$$552 = \frac{1}{2t}\sqrt{\left(\frac{Y}{\rho}\right)} \quad ...(i)$$

Let the generated frequency be ν', when thickness of the crystal is $t/2$ and Young's modulus $4Y$. Then

$$\nu' = \frac{1}{2(t/2)}\sqrt{\left(\frac{4Y}{\rho}\right)} \quad ...(ii)$$

Dividing eq. (ii) by eq. (i), we obtain

$$\frac{\nu'}{552} = 2 \times \sqrt{4} = 2 \times 2 = 4, \quad \therefore \ \nu' = 4 \times 552 = \mathbf{2208\ Hz}$$

Ex. 9. *Determine the frequency of the first and second modes of vibration for a quartz crystal of Piez-electric oscillator. The velocity of longitudinal acoustic waves in a crystal is* 5.5×10^3 m/sec. *Thickness of quartz crystal is* 2.5 cm.

Sol. For *n*th mode, the frequency of vibration of the crystal is given by

$$\nu_n = \frac{v}{\lambda} = \frac{nv}{2l}$$

For the first or fundamental mode, $\nu_1 = \frac{v}{2l} = \frac{5.5 \times 10^3}{2 \times 0.025} = \mathbf{1.1 \times 10^5\ Hz.}$

For the second mode, $\nu_2 = \frac{2v}{2l} = \frac{v}{l} = \frac{5.5 \times 10^3}{0.025} = \mathbf{2.2 \times 10^5\ Hz.}$

Ex. 10. *An ulstrasonic beam is used to determine the thickness of a steel plate. It was noticed that the difference in two adjacent harmonic frequencies is* 50 kHz. *The velocity of sound in steel is* 5000 m/s. *Calculate the thickness of the steel plate.*

Sol. Frequency of the *n*th harmonic, $\nu_n = \frac{nv}{2l}$

Difference of two adjacent i.e., *n*th and $(n-1)$th harmonics Frequencies

$$= \frac{nv}{2l} - \frac{(n-1)v}{2l} = \frac{v}{2l} = \text{fundamental frequency}$$

Thus $\frac{v}{2l} = 50$ kHz or $\frac{5000}{2l} = 50{,}000$; $\therefore\ l = \frac{1}{20} = \mathbf{0.05\ m.}$

This is the thickness of the steel plate.

Ex. 11. *Bats emit ultrasonic waves. The shortest wavelength in air by a bat is about* 0.33 cm. *What is the highest frequency a bat can emit?* (*velocity of sound in air* = 330 m/sec) **(O.U. 2010; K.U. 2007)**

Sol. The highest frequency emitted by a bat, will correspond to the shortest wavelength. If *v* be the velocity of sound in air, then the highest frequency is

$$\nu = \frac{v}{\lambda} = \frac{330\ \text{m/sec}}{0.33 \times 10^{-2}\ \text{m}} = 10^5\ \text{Hz} = \mathbf{100\ kHz.}$$

QUESTIONS

Long Answer Questions

1. What are ultrasonics? Explain the terms magnetostriction and Piezoelectric effect. Describe a method to determine the velocity and wavelength of ultrasonic waves. **(K.U. 2007)**

2. Describe the production of ultrasonics by magnetostriction method. Mention any two applications of ultrasonics. **(S.K.U. 2005)**

3. What are ultrasonic waves? Describe the magnetostriction method of producing ultrasonics. Write any four applications of ultrasonics. **(S.U. 2014; Krishna U. 2014; V.S.U. 2014)**

4. What is a magnetostriction? Explain how it is used to produce ultrasonic waves. Write a short note on the applications of ultrasonic waves. **(S.V.U. 2014)**

5. Describe how ultrasonic waves are produced by magnetostriction method with diagram. Write any five applications. **(S.K.U. 2002)**

6. What are properties of ultrasonics? Write about the production of ultrasonics using the magnetostriction principle. **(A.N.U. 2013, 07; Y.V.U. 2014)**

7. What are *X*-cut and *Y*-cut crystals? Describe the magnetostriction method of producing ultrasonics. **(O.U. 2008)**

8. Discuss Piezo-electric method and magnetostriction method for production of ultrasonics. Discuss how wavelength of ultrasonics can be dtermined? **(K.U. 2013)**

9. Explain the properties of ultrasonic waves. Discuss piezo-electric method for the production of ultrasonic waves. **(M.G.U. 2014)**

10. What is Piezo-electric effect? Explain the production and detection of ultrasonics using Piezo-electric effect. **(A.U. 2013)**

11. Describe two methods for the production of ultrasonic waves. What are the different applications of ultrasonics?

12. Distinguish between *X*-cut and *Y*-cut crystal slabs. Which is used in Piezoelectric method and why? **(O.U. 2001)**

13. What are ultrasonics? Explain the terms magnetostriction and Piezoelectric effect? Describe how ultrasonics are produced by the method of Piezoelectric method? **(O.U. 1997)**

14. What are ultrasonics? How ultrasonics are produced using Piezoelectric effect. **(A.N.U. 2007; A.U. 2005; T.U. 2014)**

15. Write in detail the production and detection of ultrasonic waves. What are its applications? **(A.N.U. 2008, 06)**

16. What are the various methods used in the detection of ultrasonic waves? **(O.U. 2001)**

17. What are ultrasonics. Explain how the ultrasonic waves are detected. **(S.K.D.U. 2013; S.K.U. 2014, 2007, 05)**

18. How do you detect ultrasonics? What are its three important applications? **(K.U. 1992; S.K.U. 2014)**

19. What are the properties of ultrasonic waves? Explain the detection of ultrasonic waves. **(A.N.U. 2012)**

20. Discuss the various methods used for the detection of ultrasonics. Discuss the applications of ultrasonics. **(K.U. 2012)**

21. Write an essay on the production, detection and applications of ultrasonic waves. **(A.U. 2007)**

22. Describe ultrasonic interferometric method for the determination of wavelength and velocity of ultrasonics in liquids. **(O.U. 2009)**

Short Answer Questions

1. What are ultrasonics and infrasonics?

2. Explain magnetostriction. **(A.U. 2007)**

3. What are ultrasonics? Explain the terms magnetostriction and Piezoelectric effect. **(K.U. 2005)**

4. What is Piezo-electric effect? Mention two examples for Piezoelectric material. **(O.U. 2003)**

5. What are *X*-cut and *Y*-cut crystals? Explain. **(S.U. 2014; K.U. 2005)**

6. Explain the concept of Piezoelectricity. **(A.U. 2008)**

7. Write short notes on Piezoelectric effect. **(A.U. 2007; S.V.U. 2002)**

8. Explain properties of ultrasonic waves. **(S.V.U. 2007)**

9. Give two properties of ultrasonic waves. **(A.U. 2005)**

10. What are applications of ultrasonics? **(A.N.U. 2012; O.U. 2002)**

11. Explain any five applications of ultrasonic waves. **(O.U. 2013, 07; S.V.U. 2002; A.U. 2007; Y.V.U. 2014; T.U. 2014; Dr. B.U. 2013)**

12. Explain various methods used in detection of ultrasonics. **(Krishna U. 2014; A.N.U. 2013)**
13. How do you determine the velocity of ultrasonics in liquid by Sear's methods? **(R.U. 2014)**
14. What are shock waves? What is Mach number. **(O.U. 2009)**

PROBLEMS

1. A magnetostriction oscillator has a frequency of 10 kHz. If it produces sound wave of velocity 6.2 $\times 10^3$ m/s. Find the length of ferrite rod. **(S.K.U. 2005)**
[**Ans.** 0.31 m]

2. A magnetostriction oscillator has a frequency of 30 kHz. If it produces sound wave of velocity 6.2 $\times 10^3$ m/sec, calculate the length of the ferrite rod.
[**Ans.** 10.3 cm]

3. Find out the fundamental frequency of a quartz crystal 1 mm thick and of density 2.65 gm/cm^3. $C_{11} = 8.516 \times 10^{11}$ dyne/cm^2.
[**Ans.** 2.855 MHz]

4. A Piezoelectric *X*-cut quartz crystal plate has a thickness of 1.6 mm. If the velocity of propagation of sound waves along the *X*-direction is 5760 meter/sec., calculate the fundamental frequency of the crystal.
[**Ans.** 1.8 MHz]

5. Calculate the fundamental frequency of a quartz crystal of thickness 0.001 m. $Y = 7.9 \times 10^{10}$ N/m^2 and $\rho = 2650$ kg/m^3. **(S.K.U. 2001; A.N.U. 2002)**
[**Ans.** 2.73 MHz]

6. A quartz crystal of thickness 3 mm is vibrating at its natural frequency. If the Young's modulus and density of quartz are 7.9×10^{10} N/m^2 and 2,650 kg/m^3 respectively, calculate the crystal frequency for resonant vibration. **(A.N.U. 2002)**
[**Ans.** 0.91 MHz]

7. Calculate the frequency of fundamental note emitted by a Piezoelectric crystal. Use the following data :
$Y = 8 \times 10^{11}$ N/m^2, $\rho = 2.5 \times 10^3$ kg/m^3 and vibrating length = 3 mm.
(S.U. 2014; V.S.U. 2014; K.U. 2012; S.K.U. 2002; A.N.U. 2001)
[**Ans.** 0.943 MHz]

8. Calculate the capacitance to produce ultrasonic waves of 1 MHz with inductance of 2 henry.
[**Ans.** 0.0125 pF]

●

www.himpub.com

ISBN: 978-93-5433-343-9 ESP 0175